



# VCG mechanisms and efficient ex ante investments with externalities

Daniel Krähmer, Roland Strausz \*

*Free University Berlin*

Received 25 January 2006; received in revised form 9 May 2006; accepted 28 June 2006

Available online 11 December 2006

---

## Abstract

For mechanism design with independent values, we identify a subclass of Vickrey–Clarke–Groves (VCG) mechanisms that induce efficient ex ante investments even with externalities. The Vickrey second price auction does not belong to this class.

© 2006 Elsevier B.V. All rights reserved.

*Keywords:* Externalities; Ex ante investments; Efficient mechanisms; Mechanism design; Second price auction

*JEL classification:* D82

---

## 1. Introduction

For mechanism design with independent values, we identify a subclass of Vickrey–Clarke–Groves (VCG) mechanisms that induce efficient ex ante investments when investments exhibit externalities. Hence, we extend the result that there is no trade-off between ex ante efficiency and ex post efficient implementation in the independent private value setting (e.g. [Bergemann and Välimäki, 2002](#); [Arozamena and Cantillon, 2004](#)). Yet, in contrast to the case without externalities not every VCG induces efficient ex ante investments. In particular, the familiar Vickrey second price auction does not induce efficient investments and therefore does not belong to the identified subclass of VCG mechanisms. Spectral

---

\* Corresponding author. Boltzmannstrasse 20, 14195 Berlin, Germany. Tel.: +49 30 838 53685; fax: +49 30 838 54142.  
*E-mail address:* [strausz@zedat.fu-berlin.de](mailto:strausz@zedat.fu-berlin.de) (R. Strausz).

auctions for mobile phones, where entrants benefit indirectly from an incumbent’s investment in fixed-line telephone connections, are typical examples of an implementation setting that exhibits ex ante investments with externalities.

## 2. Setup

There are  $n$  players  $i = 1, \dots, n$ , a social planner, a set  $X$  of possible decisions, and for each player  $i$  a set of types  $S_i = [s_i, \bar{s}_i] \subseteq \mathbb{R}$ . Let  $S = \times_i S_i$ . There are two periods. In period 1, each player  $i$  independently and simultaneously chooses an investment  $\alpha_i \geq 0$  at a personal cost  $c_i(\alpha_i)$ . After the players have chosen their investment levels their types  $s \in S$  are realized. In particular, player  $i$ ’s type  $s_i$  is the realization of a random variable  $\tilde{s}_i$  with cdf  $F_i(s_i; \alpha)$ . Since the distribution may depend on the entire investment profile  $\alpha = (\alpha_1, \dots, \alpha_n)$ , our setup allows for investment externalities. We assume that the family  $(\tilde{s}_1, \dots, \tilde{s}_n)$  is stochastically independent for all  $\alpha$  so that we consider a framework of independent, private values. Moreover, we assume that  $\tilde{s}_i$  has a pdf  $f_i(s_i; \alpha)$  that is bounded, strictly positive, and differentiable in  $\alpha$  for all  $s_i$  and  $\alpha$ .

In period 2, a decision  $x \in X$  is implemented together with a transfer schedule  $t = (t_1, \dots, t_n)$  specifying a transfer  $t_i$  of agent  $i$  to the social planner. We assume that agents have quasi-linear utility functions that are additively separable in decision, transfer, and cost. That is, given an investment profile  $\alpha$ , a type realization  $s$ , a decision  $x$ , and a transfer  $t_i$ , agent  $i$  receives a utility

$$u_i(\alpha, s, x, t) = v_i(x, s_i) - t_i - c_i(\alpha_i).$$

As is standard, we assume that the social planner collects the transfers. Therefore, total social welfare is

$$u(\alpha, s, x, t) = \sum_i u_i(\alpha, s, x, t) + \sum_i t_i = \sum_i \{v_i(x, s_i) - c_i(\alpha_i)\}.$$

## 3. Efficiency

Given a realization of types  $s = (s_1, \dots, s_n)$ , a decision  $x^* \in X$  is *ex post efficient* if and only if

$$x^* \in \arg \max_{x \in X} \sum_i v_i(x, s_i). \tag{1}$$

We call a decision rule  $x : S \rightarrow X$  *ex post efficient* if  $x^*(s)$  satisfies (1) for every  $s \in S$ .

Agent  $i$ ’s expected utility from an investment profile  $\alpha$  and an *ex post efficient* decision rule  $x^*(s)$  is

$$U_i(\alpha) = E[v_i(x^*(\tilde{s}), \tilde{s}_i)] - c_i(\alpha_i). \tag{2}$$

An *ex ante* investment profile  $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$  is *ex ante efficient* with respect to an *ex post efficient* decision rule  $x^*(s)$  if and only if it maximizes the expected sum of utilities  $U(\alpha) = \sum_i U_i(\alpha)$ . That is,

$$\alpha^* \in \arg \max_{\alpha} U(\alpha). \tag{3}$$

A combination  $(\alpha^*, x^*(s))$  satisfying (1) and (3) is overall efficient.

Finally, we define the marginal *social contribution* of investment  $\alpha_i$  as

$$\frac{\partial}{\partial \alpha_i} U(\alpha) = \frac{\partial}{\partial \alpha_i} E \left[ \sum_j v_j(x^*(\tilde{s}), \tilde{s}_j) \right] - c'_i(\alpha_i). \quad (4)$$

We assume that the ex ante efficient level  $\alpha^*$  is interior and therefore satisfies the first and second order conditions. Hence,  $\partial U(\alpha^*)/\partial \alpha_i = 0$  for all  $i = 1, \dots, n$ .

#### 4. Private information

Now assume the players' types are private information. In this case the implementation of an ex post efficient decision rule requires the transmission of private information. We suppose that the social planner may employ some mechanism to elicit such information. Due to the revelation principle, we may restrict attention to *direct mechanisms*  $(x, t) : S \rightarrow X \times \mathbb{R}^n$  that require each player to independently submit a report  $\hat{s}_i \in S_i$  and selects allocation  $(x(\hat{s}), t(\hat{s}))$  conditional on the joint reports  $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n)$ . The mechanism  $(x, t)$  implements the decision rule  $\xi : S \rightarrow X$  in dominant strategies if  $\xi = x$  and truthtelling is a dominant strategy for each player. We are interested in mechanisms that implement an ex post efficient rule  $x^*(s)$  and induce an ex ante efficient investment profile  $\alpha^*$ . It is well known that  $x^*(s)$  is implemented in dominant strategies by a Vickrey–Clarke–Groves (VCG) mechanism  $(x, t)$  (e.g. Fudenberg and Tirole, 1991, p.271ff). These direct mechanisms are characterized by

$$x(s) = x^*(s) \text{ and } t_i(s) = - \sum_{j \neq i} v_j(x^*(s), s_j) - h_i(s_{-i}),$$

where  $h_i$  is an arbitrary function that depends only on the profile  $s_{-i}$  of types excluding  $s_i$ .

We assume that players play a Nash equilibrium in period 1 thereby anticipating that a VCG mechanism is used in period 2. Given an investment profile  $\alpha$  and a type realization  $s \in S$ , a VCG mechanism yields player  $i$  a utility of

$$u_i(s, \alpha_i; \alpha_{-i}) = v_i(x^*(s), s_i) - t(s) - c_i(\alpha_i) = \sum_j v_j(x^*(s), s_j) + h_i(s_{-i}) - c_i(\alpha_i).$$

Thus, in period 1 player  $i$ 's expected utility is  $U_i(\alpha_i; \alpha_{-i}) = E[u_i(\tilde{s}, \alpha_i; \alpha_{-i})]$ , and player  $i$ 's *private incentive* to invest is

$$\frac{\partial}{\partial \alpha_i} U_i(\alpha_i; \alpha_{-i}) = \frac{\partial}{\partial \alpha_i} E \left[ \sum_j v_j(x^*(\tilde{s}), \tilde{s}_j) \right] + \frac{\partial}{\partial \alpha_i} E[h_i(\tilde{s}_{-i})] - c'_i(\alpha_i). \quad (5)$$

Inspection reveals that the private incentive exceeds the social contribution by

$$\delta_i(\alpha_i; \alpha_{-i}) = \frac{\partial}{\partial \alpha_i} E[h_i(\tilde{s}_{-i})].$$

We refer to  $\delta_i(\alpha_i; \alpha_{-i})$  as player  $i$ 's *excess investment incentive*, because if

$$\delta_i(\alpha^*) = 0, \tag{6}$$

then the private incentive to invest (5) coincides with the social contribution of investment (4). In this case, the first order necessary conditions for a Nash equilibrium  $\partial U_i(\alpha)/\partial \alpha_i = 0$  have a solution at  $\alpha^*$ . As a result, the VCG mechanism induces efficient investments  $\alpha^*$  as an equilibrium. We call a VCG mechanism with (6) *investment efficient*.

Due to independence, the pdf of the multivariate random variable  $\tilde{s}_{-i}$  is  $f_{-i}(s_{-i}; \alpha) = \prod_{j \neq i} f_j(s_j; \alpha)$  with support  $S_{-i} = \times_{j \neq i} S_j$  and cdf  $F_{-i}(s_{-i}; \alpha) = \prod_{j \neq i} F_j(s_j; \alpha)$ . Expanding  $\delta_i$  yields:

$$\delta_i(\alpha^*) = \frac{\partial}{\partial \alpha_i} \int_{S_{-i}} h_i(s_{-i}) f_{-i}(s_{-i}; \alpha_i, \alpha_{-i}^*) ds_{-i} \Big|_{\alpha_i = \alpha_i^*} = \int_{S_{-i}} h_i(s_{-i}) \frac{\partial}{\partial \alpha_i} f_{-i}(s_{-i}; \alpha^*) ds_{-i}.$$

**Proposition 1.** *A VCG mechanism is investment efficient if*

- (i)  $\frac{\partial}{\partial \alpha_i} f_{-i}(s_{-i}; \alpha) = 0$  for all  $\alpha$ , or
- (ii)  $h_i$  is a constant.

**Proof.** (i) trivial. (ii) We have

$$\int_{S_{-i}} h_i \frac{\partial}{\partial \alpha_i} f_{-i}(s_{-i}; \alpha^*) ds_{-i} = h_i \frac{\partial}{\partial \alpha_i} [F_{-i}(\bar{s}_{-i}, \alpha) - F_{-i}(\underline{s}_{-i}, \alpha)].$$

The right hand side is zero, since  $F_{-i}(\bar{s}_{-i}, \alpha) = 1$  and  $F_{-i}(\underline{s}_{-i}, \alpha) = 0$  for all  $\alpha$ . □

Condition (i) says that player  $j$ 's investment has no effect on player  $i$ 's type distribution and describes the case without investment externalities. Thus, in the absence of externalities, any VCG mechanism is investment efficient. This is the observation made in [Bergemann and Välimäki \(2002\)](#), Theorem 1.

Condition (ii) says that even in the presence of externalities, ex ante efficiency can be achieved by simply setting  $h_i$  equal to a constant. By the appropriate choice of the constants  $h_i$ , any division of the ex ante surplus among the players can be achieved. In particular, the social planner can achieve an *ex ante* balanced budget, i.e. the sum of expected transfers to the planner is 0. Note that there are other, non-constant functions  $h_i$  that satisfy (6), but these depend on the specifics of the densities  $f_j$ . Hence, following the Wilson doctrine of “detail-free” implementation (e.g. [Dasgupta and Maskin, 2004](#)) condition (ii) defines a subclass of VCG mechanisms with the advantage of being independent of any distributional details.

The significance of Proposition 1 is partially owed to a prominent VCG mechanism, the second price auction, which is not ex ante efficient. This is illustrated in the following example.

### 5. Vickrey auction is not ex ante efficient

There are two players and a single object. Let  $X = \{x_1, x_2\}$ , where  $x_i$  means that player  $i$  gets the object. Let  $S_i = [0, 1]$  and interpret  $s_i$  as player  $i$ 's valuation of the object. Suppose only player 1 can invest. Let  $\alpha_1 \in [0, 1]$  and consider

$$F_1(s_1; \alpha_1) = s_1^{\alpha_1},$$

$$F_2(s_2; \alpha_1) = s_2^{k + \gamma \alpha_1},$$

where  $k > 0$  and are constants such that  $k + \gamma\alpha_1 \geq 0$ . That is, player 1's investment decreases ( $\gamma < 0$ ) or increases ( $\gamma > 0$ ) player 2's valuation in the sense of first order stochastic dominance. In other words, there is a negative or positive spillover from player 1 to player 2.

Recall that the Vickrey auction is a specific VCG with

$$h_i(s_{-i}) = -s_{-i}.$$

Hence, player 1's private excess incentive is

$$\delta_1(\alpha_1) = \frac{-\gamma}{(k + \gamma\alpha_1 + 1)^2}.$$

Therefore,  $\delta_1(\alpha_1) \leq 0 \Leftrightarrow \gamma \geq 0$ , and we conclude that there is too much (little) equilibrium investment when there is a negative (positive) spillover.

### **Acknowledgement**

Financial support by the DFG (German Science Foundation) under SFB/TR 15 is gratefully acknowledged. Roland Strausz acknowledges financial support under grant STR-991/1.

### **References**

- Arozamena, L., Cantillon, E., 2004. Investment incentives in procurement auctions. *Review of Economic Studies* 71, 1–18.  
 Bergemann, D., Välimäki, J., 2002. Information acquisition and efficient mechanism design. *Econometrica* 70, 1007–1033.  
 Dasgupta, P., Maskin, E., 2004. Efficient auctions. *Quarterly Journal of Economics* 115, 341–388.  
 Fudenberg, D., Tirole, J., 1991. *Game Theory*. MIT Press, Cambridge.