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A Note on a Theorem of Aumann and Dreze[1]

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Abstract: Aumann and Dreze (1974) examine sections of cores of side payments games via cores of reduced games. We study sections of cores of games without side payments.

1 Introduction

Sections of cores of TU (transferable utility) games are studied in Aumann and Dreze (1974). We study sections of cores of NTU (non-transferable utility) games, thus generalizing their result. Both analysis make a strong use of reduced games, but we point out an important difference between the two cases.

Sections of cores have been used in exhibiting geometric properties of the kernel and nucleolus of TU games (Maschler, Peleg, Shapley (1979)), as well as in a generalization of the intersection of the core and kernel for NTU assignment games (Moldovanu 1988).

For an economic interpretation, consider for example an allocation in the core of an exchange economy. Assume that the allocation to a subgroup of players is hold fixed. The question is then: How can the members of the complement trade between themselves such that the resulting new allocation is still in the core of the original economy?

The reduced game property (RGP) and the converse reduced game property (CRGP) have been used, among other, in the axiomatization of cores of TU or NTU games (Peleg (1985, 1986). The main idea is one of stability of solutions under partial implementation by subgroups of players which consider their outside options.

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2 Preliminaries

Let U be a finite set of players. A coalition is a non-empty subset of U. A TU game is a pair (N,v) where N is a coalition and v is a function which assigns to each subset S in N a real number $v(S)$. We assume $v(\phi) = 0$.

A payoff vector for N is a function $x : N \Rightarrow \mathbb{R}$, thus $\mathbb{R}^N$ is the set of all payoff vectors. x^S denotes the restriction of x to members of S.

$x(S)$ denotes the sum $\sum_{i \in S} x^i$.

Let $x, y \in \mathbb{R}^N$. We write: $x \geq y$ if $x^i \geq y^i$ for all $i \in N$; $x > y$ if $x \geq y$ and $x \neq y$; $x >> y$ if $x^i > y^i$ for all $i \in N$.

Let (N,v) be a TU game. We denote

$$X(N,v) = \{ x \mid x \in \mathbb{R}^N \text{ and } x(N) \leq v(N) \} \tag{2.1}$$

The core of (N,v), $C(N,v)$, is defined by:

$$C(N,v) = \{ x \mid x \in X(N,v) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N \} \tag{2.2}$$

Let $S \subseteq N$, a coalition, and let $x \in X(N,v)$. The reduced game with respect to S and x is the game (S,v_x) where:

$$v_x(T) = \begin{cases} 0, & \text{if } T \text{ is empty} \quad (2.3) \\ v(N) - x(N \setminus T), & \text{if } T = S \quad (2.4) \\ \max \{ v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S \}, & \text{otherwise} \quad (2.5) \end{cases}$$

Let $A \subseteq \mathbb{R}^k$. A is comprehensive if $x \in A$ and $x \geq y$ imply $y \in A$. The boundary of A is denoted by ∂A and the interior of A by A°. $\mathbb{R}^k_+$ is the restriction of $\mathbb{R}^k$ to vectors with non-negative coordinates.

An NTU game is a pair (N,V) where N is a coalition and V is a function which assigns to each coalition $S \subseteq N$ a subset $V(S)$ of $\mathbb{R}^S$, such that

$$V(S) \text{ is non-empty and comprehensive} \tag{2.6}$$

$$V(S) \cap (\{x^S\} + \mathbb{R}^S_+) \text{ is bounded for every } x^S \in \mathbb{R}^S \tag{2.7}$$

$$V(S) \text{ is closed} \tag{2.8}$$

$$\text{if } x^S, y^S \in \partial V(S) \text{ and } x^S \geq y^S \text{ then } x^S = y^S \tag{2.9}$$

Let (N,V) be an NTU game and let $x \in V(N)$. A coalition can improve upon x if there exists $y^S \in V(S)$ such that $y^S >> x^S$. x is in the core, $C(N,V)$, of (N,V) if no coalition can improve upon x.

Let $S \subseteq N$, a coalition, and let $x \in V(N)$. The reduced game with respect to S and x is the game (S,V_x), where

$$V_x(S) = \{ y^S \mid (y^S, x^{N \setminus S}) \in V(N) \}, \tag{2.10}$$

$$V_x(T) = \bigcup_{Q \subseteq N \setminus S} \{ y^T \mid (y^T, x^Q) \in V(T \cup Q) \}, \text{ if } T \subsetneq S. \tag{2.11}$$

Let Γ be a class of NTU games. A solution on Γ is a function σ which assigns to each game $(N,V) \in \Gamma$ a subset $\sigma(N,V)$ of $V(N)$.

A solution σ on Γ has the reduced game property (RGP) if it satisfies the following condition: If $(N,V) \in \Gamma$, $S \subseteq N$, $S \neq \phi$ and $x \in \sigma(N,V)$, then $(S,V_x) \in \Gamma$ and $x^S \in \sigma(S,V_x)$.

A solution σ on Γ has the converse reduced game property (CRGP) if it satisfies the following condition: If $(N,V) \in \Gamma$, $x \in V(N)$ and it is true that $(S,V_x) \in \Gamma$ and $x^S \in \sigma(S,V_x)$ for every pair $S = \{i,j\}$ with $i,j \in N$, $i \neq j$, then $x \in \sigma(N,V)$.

Finally, some words on the definitions we used: The definition of reduced games in the form of 2.10-2.11 is due to Peleg. Conditions 2.6-2.9 in the definition of NTU games are needed to ensure that the definition of a core is possible, that reduced games are indeed games and that the core has the RGP. For details the reader may consult Peleg (1985).

3 Sections of the Core

Theorem 5 in Aumann, Dreze (1974) can be reformulated as following:

Theorem 3.1: Let (N,v) be a TU game with $C(N,v) \neq \phi$, and let S be a coalition in N. Let $x = (x^S, x^{N \setminus S}) \in C(N,v)$ and let $w = (y^S, x^{N \setminus S})$.

Then $w \in C(N,v)$ if and only if $y^S \in C(S,v_x)$.

Thus, if we keep $x^{N \setminus S}$ fixed and we let x^S vary, we stay in the core if and only if x^S stays in $C(S,v_x)$. This can be interpreted as the "bargaining range" of the members of S, if the core is accepted as a solution.

The "only if" part of the Theorem is the instance of the reduced game property for the core of TU games. The core of NTU games has also the RGP and CRGP (see Peleg (1985)) so it is natural to ask whether an exact analogue of Theorem 3.1 exists for NTU games. The answer is negative!

Example 3.2: Let $N = \{1,2,3\}$, $S = \{1,2\}$. We define an NTU game on N as follows:

$$V(S) = \{(x^1,x^2) \mid 2x^1 + x^2 \leq 4\} \tag{3.1}$$

$$V(N) = \{(x^1,x^2,x^3) \mid x^1 + x^2 + x^3 \leq 4.5\} \tag{3.2}$$

$$V(T) = 0^T, \text{ for } T \neq S,N \tag{3.3}$$

It is easy to check that $z = (1.5, 1, 2)$ belongs to $C(N,V)$. We compute now the reduced game (S,V_z):

$$V_z(S) = \{(x^1,x^2) \mid x^1 + x^2 \leq 2.5\} \tag{3.4}$$

$$V_z(i) = 0^i, i = 1,2 \tag{3.5}$$

Then $y^S = (0, 2.5) \in C(S,V_z)$ but $(y^S,z^3) = (0, 2.5, 2) \notin C(N,V)$!

It is easy to check that the section of $C(N,V)$ at $x^3 = 2$ is that part of $C(S,V_z)$ where $x^1 \geq 1.5$.

For cores of NTU games we have the following:

Theorem 3.3: Let (N,V) be an NTU game with a non-empty core and let $x \in C(N,V)$. Let S be a coalition in N and let $w = (y^S,x^{N\setminus S})$. Then $w \in C(N,V)$ if and only if $y^S \in C(S,V_x) \setminus A_x(S)$ where

$$A_x(S) = \bigcup_{\substack{Q \subset N\setminus S \\ \neq}} \{y^S \mid (y^S,x^Q) \in V^\circ(S \cup Q)\} \tag{3.6}$$

Proof: The reader will remark that $C(S,V_x) \setminus A_x(S)$ depends indeed only upon $x^{N\setminus S}$. If $w = (y^S,x^{N\setminus S}) \in C(N,V)$ then, by the reduced game property of the core, $y^S \in C(S,V_w)$. (See Peleg (1985), and remark that condition 2.9 for the grand coalition is essential there for this property).

The values of (S,V_w) depend only upon $w^{N\setminus S} = x^{N\setminus S}$ and therefore $y^S \in C(S,V_x)$. If $y^S \in A_x(S)$ then $(y^S,x^Q) \in V^\circ(S \cup Q)$ for a certain $Q \subset N\setminus S$. Then $S \cup Q$ can improve upon w, a contradiction to $w \in C(N,V)$.

For the converse part let $y^S \in C(S,V_x) \setminus A_x(S)$. Then $y^S \in V_x(S)$ and, by the definition of reduced games, $w = (y^S,x^{N\setminus S}) \in V(N)$. $y^S \in C(S,V_x)$ implies that $y^T \notin V_x^\circ(T)$ for all $T \subseteq S$, and this means:

i) $(y^T,x^Q) \notin V^\circ(T \cup Q)$ for all $T \subseteq S$ and $Q \underset{\neq}{\subset} N\setminus S$.

ii) There is no u^S with $u^S >> y^S$ and $(u^S,x^{N\setminus S}) \in V(N)$. (Remark the use of 2.9)

Assume on the contrary that $w \notin C(N,V)$. Then, there exist a coalition $L \subseteq N$ and $z^L \in V(L)$ with $z^L >> w^L$. If $L \cap S \neq S$ we get a contradiction to i), so L must contain S. If $L = N$ then, by comprehensiveness (2.6), $(z^S, x^{N \setminus S}) \in V(N)$. Because $z^S >> y^S$ we get a contradiction to ii).

Thus, if $w \notin C(N,V)$, we showed that we have a coalition of the form $L = S \cup P$ where $P \neq N \setminus S$, and $z^{S \cup P} \in V(S \cup P)$ with $z^{S \cup P} >> w^{S \cup P}$. By comprehensiveness $w^{S \cup P} \in V(S \cup P)$. It is clear that $w^{S \cup P} = (y^S, x^P) \in V^o(S \cup P)$ and we obtain a contradiction to the choice of y^S in $C(S,V_x) \setminus A_x(S)$. ■

One may ask why the argument of Aumann and Dreze for TU games does not work for NTU games. The crucial point is as follows: For a TU game (N,v) and $x \in C(N,v)$ we have in a reduced game (S,v_x) that $v_x(S) = x(S)$. Because $x \in C(N,v)$ we get $x(S) \geq v(S)$ and thus $v(S) \leq v_x(S)$. It we transform this TU game into a NTU game, in the usual manner, we get $V(S) \subseteq V_x(S)$. For a general NTU game (N,V) and $x \in C(N,V)$ it is no longer true that in (S,V_x) we have always $V(S) \subseteq V_x(S)$. ($V(R) \subseteq V_x(R)$ for $R \underset{\neq}{\subset} S$!).

For example, in Example 3.1, the correction which has to be made, namely $x^1 \geq 1.5$, is exactly for that part of $V(S)$ which is not included in $V_x(S)$.

In some special cases one can still have exact analogues of Theorem 3.1 for NTU games: An NTU game (N,V) is said to be decomposable if there is partition of N, $(B_1,B_2,...,B_k)$, such that

$$V(S) = \prod_{i=1}^{k} V(S \cap B_i) \text{ , for all } S \subseteq N \tag{3.7}$$

Proposition 3.4: Let (N,V) be a decomposable game with partition $(B_1,...,B_k)$ and non-empty core. Let $x \in C(N,V)$ and $w = (y^{Bj}, x^{N \setminus Bj})$. Then $w \in C(N,V)$ if and only if $y^{Bj} \in C(B_j, V_x)$.
The proof is left to the reader.

Of course, one may try to find another plausible definition of reduced games for the NTU case, such that an exact analogue of Theorem 3.1 will hold. My efforts in this direction (together with Terje Lensberg) led invariably to a violation of the reduced game property of the core.

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