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# Cyclical Delay in Bargaining with Externalities

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Externalities between buyers are shown to induce delays in negotiations between a seller and several buyers. Delays arise in a perfect and complete information setting with random matching even when there is no deadline. While with a deadline we identify delays both for positive and negative externalities, without a deadline we find that (1) when externalities are positive, there exists no SPNE in pure strategies with bounded recall that exhibits delay; (2) when externalities are negative, it may happen that all SPNE with bounded recall have the property that long periods of waiting alternate with short periods of activity: This is the cyclical delay phenomenon.

## 1. INTRODUCTION

We consider negotiations about the sale of an indivisible good to one of several potential buyers. The typical negotiations we have in mind deal with the privatization of a publicly-owned firm, or the sale of an innovation in an oligopolistic market. Throughout the paper, we assume that there are identity-dependent external effects between the buyers. That is, if buyer $i$ acquires the good, buyer $j$ obtains a payoff that is dependent on the identities of both $i$ and $j$.

The presence of identity-dependent externalities is a widespread feature in a variety of economic, political and social conflict situations. As an illustration, consider the following story taken from *The Economist*, April 4th 1992: South Korea plans to build a high-speed train network between Seoul and Pusan. The firms competing to obtain the contract are: A Japanese consortium (headed by Mitsubishi Corporation), Germany's Siemens, and GEC Alsthom (a joint venture between the French Alcatel Alsthom and Britain's General Electric Company, builders of the Train à Grande Vitesse). South Korea insists that the contract winner will transfer as much as possible technology to local firms. Hence, the winning company will help create a low-cost competitor in the market for fast trains. The identity of the winner matters a lot because the three firms do not have the same technology.

Another example that fits well in our framework is the negotiation leading to the sale of intangible property (say, a patent). The situation we consider here is that of a single licensee (the actual buyer of the object) causing external effects to all other competitors. In most of the literature on patent licensing, externalities are assumed to be symmetric, since they are only dependent on the number of licensees (see Katz and Shapiro (1986)). In our framework symmetry is not required.

There are two types of delay in theoretical models of bargaining. In an incomplete information setting, delay serves as a signalling device (see Kennan and Wilson (1993) for a detailed recent survey of this literature). In a complete information setting, studies that explain delay are rare. In finite-horizon frameworks, delay has been observed in a two-period model with simultaneous moves (Dekel (1990)), in a model of pretrial negotiation (Spier (1992)), in a model where the played game changes along the play path (Fershtman and Seidman (1993)), and in a model where the transmission time for offers is random (Ma and Manove (1993)). (See also Jehiel and Moldovanu (1995) which is devoted to the characterization of equilibria in the finite-horizon, undiscounted bargaining game with negative externalities.) In an infinite-horizon model that is a variation of Rubinstein's (1982) game with multiple equilibria, Haller and Holden (1990) and Fernandez and Glazer (1991) have exhibited equilibria with delayed agreements.

The main contribution of this paper is to show that, in a complete and perfect information framework, the mere presence of identity-dependent externalities between potential buyers may cause delays in negotiations even when there is no imposed deadline.

We consider the following situation: A seller owns an indivisible object to be sold to one of several potential buyers who may have different valuations for the good. At each stage, the seller meets randomly one of the potential buyers. Then the seller proposes a price that may either be accepted or rejected by the buyer. In case of acceptance, the negotiation ends. In the case of rejection, the negotiation proceeds to the next stage which has the same structure, except in the finite-horizon framework where, if it is the deadline, the seller keeps the good. The only departure from standard negotiation games is that if, say, buyer $i$ gets the good, then buyer $j$ obtains a payoff that is dependent both on $i$ and $j$. That is, buyer $j$ incurs an externality from buyer $i$ having the good.[1] All utilities (including external effects) are assumed to be time-discounted, possibly in different ways for the seller and the buyers. We will focus on the situation where the seller and the buyers are patient. Obviously, our specification of bargaining and of the matching technology is special, but it should become clear that our insights with respect to delay would continue to hold in any other bargaining model, provided that there is some random element[2] in the matching process, and that the seller is allowed to make the proposals with a strictly positive probability.

We first briefly consider the finite-horizon framework with an exogenous deadline. We observe that, with positive externalities, delay may arise naturally: Since at the end of the negotiation game, buyers are sure that someone will eventually get the good, at earlier stages, they prefer not to buy. The object is sold only a short time before the deadline. More surprisingly, we also find delays in the exactly opposite situation, i.e. with negative externalities. Such delays require at least three buyers. The point is that with more than two buyers, negative externalities can play the role of positive externalities. Indeed, imagine there are two buyers, say 1 and 2, to whom the seller would be happy to sell the good, and suppose these buyers are not very afraid of each other, but are very afraid of buyer 3, say. Then, buyer 1 may well perceive the fact that buyer 2 gets the good as a positive externality as compared to the case where buyer 3 gets the good: An ensuing "war of attrition" between buyers 1 and 2 yields delay.

1. With positive externalities, the above negotiation game corresponds to a situation where a single individual has to pay for a public good. Bliss and Nalebuff (1984) addressed such a question in an incomplete information setting.

2. The randomness in the matching may be related to anonymity ideas which are sometimes required for public sales of the privatization type mentioned above (e.g. the government (the seller) may not be allowed to discriminate between the potential buyers of a public firm).

We believe that economic settings where delays with negative externalities may appear are frequent and plausible. For example, assume that the good to be sold is a cost-reducing innovation whose effect is similar to a technology that firms 1 and 2 possess already. Firm 3 is a less efficient competitor, and the innovation would make firm 3 as efficient as the two other firms. One can show that the delay phenomenon arises if firm 3 is, originally, not too inefficient relative to firms 1 and 2.

We next consider an infinite-horizon framework. We restrict our attention to Subgame-Perfect Nash Equilibria that use pure strategies with bounded recall. The idea is that the memory capacity of the players is finite, and therefore their behaviour may only depend on what happened in the last $M$ stages (we do not restrict $M$ as long as it is finite). We first observe that, if there is any delay in the infinite-horizon framework, periods of activity must necessarily alternate with periods of waiting. This happens because, if the transaction were delayed for a very long time (*a fortiori* for ever), the seller would be happy to sell at a rather low price so as to avoid discounting; since the buyers are happy with buying at a low price, an immediate transaction occurs: It follows that long periods of waiting generate periods of activity.

Assuming that the seller is sufficiently patient relative to the buyers, we show that with positive externalities, no SPNE with bounded recall exhibits delay. The only equilibria in pure strategies with bounded recall are stationary equilibria of the following type: There is a buyer—whom we call the effective buyer—such that, at every stage, if the seller meets this buyer, he makes a proposal that is accepted, and if he meets any other buyer, he prefers to wait (until he finds the effective buyer).[3] The intuition for the disappearance of delays in the infinite-horizon framework is as follows. Delays with positive externalities are sustained by the belief of each buyer $i$ that some buyer $j$ other than $i$ will get the good if he (buyer $i$) rejects the offer. While such beliefs may be consistent when there is a deadline, it happens that, in the long run, such a belief is not consistent for at least one buyer (the effective buyer in the above equilibrium). That buyer should subsequently accept to pay a price close to his valuation, since he has no hope that someone else will get the good. The seller is happy to sell (resulting in no delay) because, as externalities are positive, the seller cannot expect a higher price than the valuation of some buyer.

Surprisingly, we find that the disappearance of delays in the infinite-horizon framework does not apply to the case of negative externalities. For parameter values fitting the oligopoly example outlined above, we show that, whenever the seller is sufficiently patient relative to the buyers, *all* SPNE with bounded recall have the feature that long periods of waiting alternate with short periods of activity: This is referred to as "cyclical delay". A consequence is that the presence of an imposed deadline is not essential for the derivation of delay. The features of the example that drive cyclical delays are: (1) Some negative externalities play the role of positive externalities, and (2) The buyer who exerts the worst external effect has the highest valuation (gross of external effects). The second feature guarantees that the buyer who exerts the worst external effect can credibly be an effective potential buyer at some stages. The first feature guarantees that, because that buyer is an effective buyer at some stages, there is a delay phenomenon of the finite-horizon type prior to such stages (as a consequence of a "war of attrition"). It will be clear that cyclical delays will arise for any situation such that the above two features are met.

3. When the buyer with maximum valuation is the effective buyer, say, such stationary strategies clearly define a SPNE irrespective of whether the seller is more or less patient than the buyers, provided the seller and the buyer are sufficiently patient. Our finding is stronger though, since it shows that all equilibria are of this type when the seller is sufficiently patient relative to the buyers.

The results given so far are summarized in the following table:

| | Deadline | No Deadline |
|---|---|---|
| **Positive externalities** | **Delay** | **No Delay (Prop. 4.1)** |
| No externalities | No Delay | No Delay |
| **Negative externalities** | **Delay (Example 3.1)** | **Cyclical Delay (Result 4.2)** |

The remainder of the paper is organized as follows. In Section 2 we describe the model. In Section 3 we consider the finite-horizon model. In Section 4 we consider the infinite-horizon model, and discuss the extension of the above described results to more general patience structures. In particular, we prove that the cyclical delay phenomenon may continue to hold even when the sell and the buyers are equally patient. Finally, in Section 5 we present some concluding remarks.

## 2. THE NEGOTIATION PROCEDURE

We consider a market consisting of $N$, $N \geqq 2$, buyers, and a seller $S$. The seller owns one unit of an indivisible good. We normalize the utility functions of the agents in such a way that their utility when no trade takes place is equal to zero. The buyers are denoted by $i$, $j$, etc. ..., $1 \leqq i, j \leqq N$. If buyer $i$ owns the indivisible good, then his utility is given by a valuation $\pi_i$. When one buyer acquires the indivisible good, all other buyers are subject to an external effect. The utility of $j$ if $i$ owns the good is given by $-\alpha_{ij}$. If $\alpha_{ij} \geqq$(resp. $\leqq$)0, then buyer $j$ suffers a negative (resp. positive) externality from buyer $i$.

The indivisible good is to be sold through a bargaining procedure which has, potentially, several stages. The finite-horizon case corresponds to the presence of an exogenous deadline, whereas the infinite-horizon allows the parties to bargain for ever. From a practical point of view, deadlines are rather frequent in negotiations: They correspond to changes of regime in the relationship between the parties. However, the finite-horizon framework is appropriate only if, from the beginning of the bargaining procedure, the parties do clearly perceive the presence of a deadline. Otherwise, the infinite-horizon framework is more appropriate (see Osborne–Rubinstein (1990, page 54)).

We now describe the bargaining procedure. Stages are indexed by $k = 1, 2, \ldots$. At the beginning of any stage, the seller meets randomly one of the buyers, and all buyers have the same probability $(1/N)$ of meeting the seller. If $i$ and $S$ meet, then $S$ proposes a transaction at price $p$. The price $p$ belongs to the interval $[0, P]$. The upper bound $P$ satisfies the conditions $P > \text{Max}_{i,j}(\pi_i + \alpha_{ji})$ and $P > \text{Max}_i(\pi_i)$. Hence $P$ is larger than any price that will be ever acceptable to any buyer. By proposing an unacceptable price, say $P$, the seller basically "waits", e.g. decides to postpone the transaction. Consequently the modified version of the game where the seller could either choose to wait or propose a transaction price would not effectively enlarge the action space of the seller. We may, therefore, without loss of generality, restrict the action space of the seller to price proposals.

If $S$ proposes $p$, then $i$ can either accept or reject the proposal. If $i$ accepts, then he obtains the good, pays price $p$ to the seller, and the game ends. Evaluated at the time of the sale, the utility of the seller is given by $p$, the utility of the buyer is $\pi_i - p$, and that of buyer $j$ is $-\alpha_{ij}$, for all $j \neq i$. If $i$ refuses the proposal, or if $S$ has chosen to wait, then there are two possibilities. In the finite-horizon model, if the game has already reached the last stage, then the game ends (with no trade taking place); otherwise the game continues to the next stage. In the infinite-horizon model, when $i$ refuses the proposal, the game can only continue to the next stage. In both cases, the following stage, if any, has the same structure as described above. If the good is never sold, then the utilities of all agents are zero (see the normalization above).

The seller and the buyers discount the future. We denote by $\delta_S$ and $\delta_B$ the discount factors of the seller and the buyers, respectively.[4] External effects are discounted as well, that is, if $i$ is to get the good, and if this is harmful to $j$ (negative externality), the later the better for $j$. (The converse is true with positive externalities.)

The infinite-horizon game where the seller has discount factor $\delta_S$, and the buyers have discount factor $\delta_B$ is denoted by $\Gamma_{\delta_S,\delta_B}$. The finite-horizon $T$-stage game is denoted by $\Gamma_{T,\delta_S,\delta_B}$.

In the following analysis, we restrict attention to parameters $(\pi_i, \alpha_{ij})_{i,j}$ that are generic in the sense that there are no rational numbers, $(x_i, y_{ij})_{i,j,j\neq i}$ such that: $\sum_{1\leqq i\leqq N}(x_i\cdot\pi_i)+\sum_{1\leqq i,j\leqq N,i\neq j}(y_{ij}\cdot\alpha_{ij})=0$, and at least one scalar in $\{x_i, y_{ij}\}_{i,j,j\neq i}$ differs from zero.

*Remark.* We have assumed that only the seller can make price proposals. A more symmetric assumption would allow the seller and the (selected) buyer to make the proposal, say each with probability half. We have not chosen the symmetric specification because it would significantly complicate the notation without affecting the nature of our results (w.r.t. delay). More generally, the results would remain qualitatively the same as long as there is some random element[5] in the matching process, and the seller makes proposals with some strictly positive probability.

## 3. THE FINITE-HORIZON GAME

There are $T$ stages, and we will mainly be concerned with some asymptotic properties of equilibrium strategies when the number of stages, $T$, tends to infinity, and the players get infinitely patient, i.e. $\delta_S$, $\delta_B$ tend to one. By a larger number of stages we do not mean that the duration of the negotiation gets longer. Instead, we mean that the rhythm of the negotiation accelerates, or, more precisely, that the pace at which one switches from one negotiation round to the next one increases.

We first introduce some notation. Since we consider generic values of the parameters $(\pi_i, \alpha_{ij})_{i,j}$, it can be shown that for $\delta_S$ and $\delta_B$ sufficiently close to one, the $T$-stage game, $\Gamma_{T,\delta_S,\delta_B}$, has a unique Subgame-Perfect Nash Equilibrium in pure strategies (SPNE) denoted by $\sigma_{T,\delta_s,\delta_B}$. Given $\sigma_{T,\delta_S,\delta_B}$, we denote by $I^k$ the set of potential buyers at stage $k$. Buyer $i$ belongs to $I^k$ if, whenever $S$ and $i$ meet at stage $k$, in equilibrium $S$ makes a proposal $p$ such that $i$ accepts it and buys the good. We denote by $C^k$ the cardinality of the set $I^k$. Let $p_i^k$ be the maximum price that buyer $i$ would be willing to pay at stage $k$, and let $p_S^k$ be the minimum price that the seller is ready to offer at stage $k$. We denote by $V_i^k$ and $V_S^k$ the expected payoff to buyer $i$ and to the seller, respectively, at stage $k$, before nature has selected whom $S$ meets at that stage. To summarise, if the seller meets buyer $i\in I^k$ at stage $k$, in equilibrium the seller proposes the price $p_i^k$ and buyer $i$ accepts the

4. We do not require that the seller and the buyers have the same discount factors because they may be differently patient, or they may have different accesses to the credit market, which is quite plausible in the privatization context where the seller is the government.

5. What really matters is that the buyers and the seller *perceive* there is some randomness in the matching.

offer. Otherwise, the seller prefers to wait for the following stage. We obtain the following recursive formulae:

For all $k \leqq T$,

$$V_S^k = \frac{1}{N}\left[\sum_{i \in I^k} p_i^k + (N - C^k)\delta_S V_S^{k+1}\right] \tag{3.1}$$

$$V_i^k = \frac{1}{N}\left[\pi_i - p_i^k + \sum_{j \in I^k, j \neq i} (-\alpha_{ji}) + (N - C^k)\delta_B V_i^{k+1}\right] \quad \text{if } i \in I^k \tag{3.2}$$

$$V_i^k = \frac{1}{N}\left[\sum_{j \in I^k} (-\alpha_{ji}) + (N - C^k)\delta_B V_i^{k+1}\right] \quad \text{if } i \notin I^k \tag{3.3}$$

$$i \in I^k \Leftrightarrow p_i^k > p_S^k \tag{3.4}$$

$$p_S^k = \delta_S V_S^{k+1} \tag{3.5}$$

$$\pi_i - p_i^k = \delta_B V_i^{k+1} \tag{3.6}$$

where we set (by convention):[6]

$$V_S^{T=1} = V_i^{T+1} = 0. \tag{3.7}$$

Condition 3.4 states that the seller makes an acceptable offer to buyer $i$ at stage $k$ if and only if the maximum price buyer $i$ is ready to accept is no less than the minimum price the seller is ready to offer. (Because the parameters are generic, situations where $p_i^k = p_S^k$ can be ruled out by induction.) Condition 3.5 states that the seller is indifferent between selling the object at the minimum price he is ready to offer at stage $k$ and waiting for the next stage (i.e. making an outrageous offer). Condition 3.6 sets the limit for what a buyer is prepared to pay at stage $k$: This maximum price renders the buyer indifferent between buying at stage $k$ and refusing the offer. Finally, condition 3.7 states that if the game ends without the good changing hands, the payoff to all players is zero. We can now define delay:

*Definition* (*Delay*). We say that there is delay at stage $k$ if the set of potential buyers at stage $k$, $I^k$, is empty.

In other words, delay at stage $k$ means that whoever the seller meets at that stage, in equilibrium he prefers to wait (i.e. make an outrageous offer). It should be clear that if, at some stage, the seller prefers to wait when he meets, say, buyer 1, but is happy to sell the good when he meets, say, buyer 2, then this is not a delay, but rather search for the best buyer.

Consider first the case where there is no discounting: $\delta_S = \delta_B = 1$. We observe that delays may arise when externalities are all positive. Assume that all buyers have the same valuation $\pi$ and all externalities are strictly positive. We obtain that: $I^T = \{1, 2, \ldots, N\}$, and, for $k < T$, there is a delay at stage $k$, i.e. $I^k = \varnothing$. The reason is as follows: In the last stage, buyers are ready to pay their valuation price, i.e. $\pi$, while in the stage just before they are not ready to pay that much since, in case, say, buyer 1 rejects the offer, some other buyer will get the good with probability $(1 - (1/N))$. This is good to buyer 1 because externalities are positive. By waiting until the last period, the seller can guarantee himself

6. This is needed for the computations at stage $T$.

at least the price $\pi$. Since waiting is not costly, the seller prefers to wait, and delay results at stage $T-1$. Moreover, delay at stage $T-1$ implies delay at every earlier stage, since there is no discounting. It should be clear that delays will also arise for parameters $(\pi_i, \alpha_{ij})_{i,j}$ in a neighbourhood of those just considered.[7]

The delay arising with positive externalities is essentially due to the fact that the maximal price the buyers are ready to pay is an increasing function of $k$. One might then conjecture that there is no delay with negative externalities. Indeed, one might expect that in that case the maximal price a buyer is ready to pay at stage $k$ decreases with $k$: Every buyer obviously prefers that nobody gets the good (rather than some other buyer), and, as the number of stages that remains to be played increases, the probability that some other buyer gets the good increases as well. However, it is not true that with negative externalities, there is no delay. The point is that even though the probability at stage $k$ that some other buyer gets the good decreases on average as $k$ increases, it may be the case that the probability that a given buyer, say buyer $j$, gets the good increases. If buyer $j$ is very harmful to buyer $i$, then the maximal price $i$ is ready to pay may increase. In such a case, the negative externality caused to $i$ by $k \neq j$ plays the role of positive externality. We now exhibit an example of this type (three buyers at least are required).

*Example* 3.1. Let $N=3$, $\pi_1=\pi_2=\pi$, $\pi_3=\pi+\varepsilon$, $\alpha_{12}=\alpha_{21}=\alpha_{13}=\alpha_{23}=0$, $\alpha_{31}=\alpha_{32}=\alpha$, where $0<\varepsilon$, and $3\varepsilon<\alpha$. We show that for $T>3$, delay occurs at all stages $k<T-2$. Note first that the maximal price 3 is ready to pay is always $\pi+\varepsilon$, while 1 and 2 would pay a premium in the order of magnitude of $\alpha$ to avoid 3 getting the object. Since $3\varepsilon<\alpha$, it is readily verified that $I^T=I^{T-1}=\{1, 2, 3\}$. Moreover, using the recursive formulae we get: $V_S^{T-1}=\pi+(\varepsilon/3)+(2\alpha/9)$, and $p_3^{T-2}=\pi+\varepsilon<V_S^{T-1}$ for $3\varepsilon<\alpha, p_1^{T-2}=p_2^{T-2}=\pi+(\alpha/3)>V_S^{T-1}$. Hence, $I^{T-2}=\{1, 2\}$. This yields: $V_S^{T-2}=\pi+((10\alpha+3\varepsilon)/27)$ and $p_1^{T-3}=p_2^{T-3}=\pi+(8\alpha/27)<V_S^{T-2}, p_3^{T-3}=\pi+\varepsilon<V_S^{T-2}$ so that $I^{T-3}=\varnothing$. At stage $T-3$, buyers 1 and 2 each hope that the other will get stuck at stage $T-2$, and therefore they are not prepared to pay enough (from the viewpoint of the seller). This implies that delay occurs at stage $T-3$ and at all earlier stages. Note that we obtain the same structure of $I^k$ in a neighbourhood of the above valuation prices and externalities.[8]

A plausible economic situation that supports parameter values as in Example 3.1 follows. There are three competitors in a market. The object to be sold is an innovation that permits a cost reduction to firm 3 only. Firms 1 and 2 already possess a similar advanced technology. The valuation of firms 1 and 2 is thus $\pi=0$,[9] while firm 3 has a strictly positive valuation $\varepsilon>0$. If firms 1 or 2 acquire the object, this has no effect, which results in $\alpha_{12}=\alpha_{21}=\alpha_{13}=\alpha_{23}=0$. If firm 3 acquires the innovation, 3 becomes a more efficient competitor and $\alpha_{31}=\alpha_{32}=\alpha>0$. By restricting attention to constant marginal cost technologies and linear demands, one can show that, if the original technology of firm 3 is not too inefficient relative to the new technology to be sold, then the condition $3\varepsilon<\alpha$ of Example 3.1 is met. Hence, it is only when firm 3 is a significant (even though less efficient) competitor that the delay phenomenon arises. The example just outlined is

7. When the valuations are all different, delay occurs only if the externalities are sufficiently large, i.e. for $\alpha_{ij}<\alpha^*<0$ whatever $i, j$. By upper-hemicontinuity w.r.t. $(\pi_i, \alpha_{ij})_{i,j}$ of the SPNE correspondence, this is consistent with the result that when there are no externalities there is no delay (see below).

8. If at each stage the buyer could make the offer with probability half, we would continue to find exactly the same sets of potential buyers, provided the valuation $\pi$ is sufficiently small relative to the externality term $\alpha$.

9. If firms 1 and 2 did not possess an exactly similar advanced technology, then $\pi$ would be positive but small.

very special, but it should be clear that delay will persist in all situations that have the same qualitative structure.

Several comments are in order. Note first that the above analysis extends to the case where the players discount the future but are patient. Indeed for a given number $T$ of stages, and for discount factors close to one, the SPNE correspondence is (generically) a continuous function of $\delta_S$ and $\delta_B$. To see this it is enough to show that the sets of potential buyers $I^k$, $k=1,\ldots,T$, associated with $\Gamma_{T,\delta_S,\delta_B}$ remain the same whatever $\delta_S$ and $\delta_B$ in a neighbourhood of 1. The result then follows because conditional on $I^k$, conditions (3.1)–(3.2)–(3.3)–(3.5)–(3.6) define continuous mappings of $\delta_S$ and $\delta_B$. The sets $I^k$ remain the same (for generic situations) because they are characterized by strict inequalities (see 3.4), and because, as the discount factors approach one, these inequalities remain the same. Consequently, in the above examples, delay will persist when players discount the future but are patient.

The second comment is that, when there are no externalities, our negotiation procedure leads asymptotically to the Walrasian outcome as $T$ tends to infinity, and the discount factors tend to one. Therefore, when there are no externalities, there is no delay. To see this, we construct the SPNE of $\Gamma_{T,1,1}$ when $\alpha_{ij}=0$, for all $1\leqq i, j\leqq N$. $I^T=\{1,2,\ldots,N\}$, $i\in I^{T-1}$ if and only if $\pi_i>1/N\sum_j \pi_j$, which is satisfied at least for the buyer with maximum valuation $\pi_i$. Continuing in this way, it is readily verified that, whatever $T$, there is no delay at any stage (because the buyer with maximum valuation necessarily belongs to $I^k$ for all $k$). Moreover, since all buyers $j$, $j\neq \text{Arg max}_i\, \pi_i$ must eventually drop out of $I^k$, there exists $k^*$ such that at any stage $k<T-k^*$, $I^k$ consists of the buyer with maximum valuation (this buyer is uniquely defined for generic situations). Since $k^*$ is constructed irrespective of $T$, it follows that in the limit as $T$ tends to infinity, the probability that the good is sold to the buyer with maximum valuation tends to one: This is the Walrasian outcome. This type of equilibrium where, at almost all stages $k$, $I^k$ consists of one buyer only is referred to as the *well-defined* buyer case.

We have identified above two types of equilibria: (1) the well-defined-buyer-type equilibrium, and (2) the delay-type equilibrium. In Jehiel and Moldovanu (1995), we have shown that, when there is no discounting and an exogenous deadline prevails, the SPNE in generic situations is necessarily of one of these two types. A consequence that will repeatedly be used in the following analysis is that, for discount factors sufficiently close to one, at arbitrary many stages before the deadline, the equilibrium sets $I^k$ will have one of the two possible structures just mentioned.

## 4. THE INFINITE-HORIZON GAME

Finding all SPNE of $\Gamma_{\delta_S,\delta_B}$ is a very complex problem. We restrict our attention to the class of strategies—denoted hereafter $\mathscr{C}$—that are (1) pure, and (2) with bounded recall (we do not restrict the memory capacities as long as they are finite). A strategy with bounded recall and memory capacity $M$ is such that the action prescribed by the strategy at any stage $t>M$ may not depend on those actions taken at stages $t'<t-M$ (see, for example, Kalai and Stanford (1988, Section 5)). The underlying idea for the bounded memory requirement is that the players do not recall what happened in remote stages, and therefore the strategy cannot be made contingent on what happened in those stages.

Apart from limitations of memory capacities, the literature on repeated games has also considered the notion of bounded complexity. We follow Kalai and Stanford (1988) by defining the complexity of a strategy as the cardinality of the set of strategies induced by it in all subgames. It may be argued that more plausible strategies are those with bounded recall and bounded complexity. In our repeated alternate-move game it happens that we are able to characterize all SPNE in pure strategies with bounded recall whether with bounded, or unbounded complexity. Therefore we will *a fortiori* consider the SPNE which, in addition to requiring bounded recall, also require bounded complexity (see Subsection 4.3).

A strategy profile in $\mathscr{C}$ is described as follows. Consider stage $k$, prior to the matching between the seller and one of the buyers. Assume that the object has not been sold yet. The history is determined by (1) the sequence of matchings between the seller and the buyers at every stage $k'$, $k'<k$, and (2) for each matching at stage $k'$, $k'<k$, a price proposal made by the seller (that has been rejected). Players have bounded recall with memory capacity $M$,[10] which means that they can remember only the last $M$ matchings and price proposals at stages $k-1, \ldots, k-M$. The associated truncated history is denoted by $h_M$. The players are assumed to identify the current time period. The behaviour strategy of the seller at stage $k$ is also contingent on the identity of the buyer $i$ that $S$ meets at that stage: It specifies a price proposal which is a function of $h_M$, the identity of the buyer $i$ and the time period $k$. The behaviour strategy of buyer $i$ who is met at stage $k$ is an acceptance decision rule which is a function of $h_M$, the price proposal $p$ and the time period $k$. Given the strategies and the truncated history $h_M$, we define $p_i^k(h_M)$ as the maximum price that $i$ is ready to accept at stage $k$. We define $p_{S,i}^k(h_M)$ as the minimum price that the seller is ready to offer to buyer $i$ at stage $k$ given $h_M$. If $S$ meets $i$ at stage $k$ and $p_i^k(h_M) \geqq p_{S,i}^k(h_M)$, then the seller proposes the price $p_i^k(h_M)$,[11] and this is accepted. Otherwise, the seller makes a proposal that is rejected. In Appendix A we show the following:

**Lemma 4.1.** *Let $\sigma$ be a* SPNE *in pure strategies with bounded memory capacity $M$. At each stage $k$, the behaviour strategy of the seller depends only on the current time $k$, and on the identity of the buyer met. The behaviour strategies of the buyers depend only on the current time $k$ and the current price offer.*

In other words, the SPNE with strategies in $\mathscr{C}$ are such that the behaviour strategies at stage $k$ are independent of $h_M$. The intuition for Lemma 4.1 is that if, at stage $k$, the strategies depend on the last $M$ actions, then necessarily at stage $k-1$, they can depend only on the last $M-1$ actions: One gets the desired history-independence result by induction.[12]

We can thus restrict our attention to the class of history-independent equilibria in pure strategies of $\Gamma_{\delta_S,\delta_B}$. This class is denoted by $\mathscr{E}(\Gamma_{\delta_S,\delta_B})$. Let $\sigma_{\delta_S,\delta_B} \in \mathscr{E}(\Gamma_{\delta_S,\delta_B})$. The set of potential buyers at stage $k$ (associated with $\sigma_{\delta_S,\delta_B}$) is denoted by $I^k$. The cardinality of the set $I^k$ is denoted by $C^k$. The maximum price that buyer $i$ would accept to pay at stage $k$ depends only on $k$: it is denoted by $p_i^k$. The minimum price that the seller is ready to offer at stage $k$ is the expected payoff he would get by waiting one more period. Hence, it is a function of $k$ only: It is denoted by $p_S^k$. Finally, $V_i^k$ and $V_S^k$ denote the expected payoffs to buyer $i$ and to the seller, respectively, at stage $k$, before nature has selected whom $S$ meets at that stage. Because $\sigma_{\delta_S,\delta_B}$ is a SPNE, the expressions for $p_i^{k+1}, p_S^{k+1}, V_i^{k+1}, V_S^{k+1}$ can be computed from $p_i^k, p_S^k, V_i^k, V_S^k$ using the recursive formulae 3.1 to 3.6 introduced in Section 3.

10. If the players do not have the same memory capacity, then $M$ stands for the maximum memory capacity among all players.

11. For simplicity, we assume that in case of indifferences the seller sells the good.

12. A similar argument appears in Jéhiel (1995).

Finally, we will say that $\sigma_{\delta_S,\delta_B} \in \mathscr{E}(\Gamma_{\delta_S,\delta_B})$ displays delay if there is a stage $k$ such that the associated set of potential buyers is empty, i.e. $I^k = \emptyset$.

### 4.1. *No delay with positive externalities*

The main result of this subsection is that when all externalities are positive, and when the seller is sufficiently more patient than the buyers, no equilibrium displays delay. That is, at every stage $k$, there is a buyer who obtains an acceptable offer if met. Formally,

**Proposition 4.1.** *Consider a generic situation with positive externalities. There exists a discount factor* $\delta_B^* \in (0, 1)$ *and a function* $\delta_S^*(\cdot): (0, 1) \to (0, 1)$ *such that no* $\sigma_{\delta_S,\delta_B} \in \mathscr{E}(\Gamma_{\delta_S,\delta_B})$ *with* $\delta_B \geqq \delta_B^*$, $\delta_S \geqq \delta_S^*(\delta_B)$ *displays delay, i.e.* $\forall k$, $I^k \neq \emptyset$.

*Proof.* See Appendix B. ||

We now give the intuition behind Proposition 4.1. Assume first, by contradiction, that there is an equilibrium that displays delay at infinitely many stages. Let $k''$ be a stage of delay. Using conditions 3.1 to 3.6, we can construct the equilibrium values of $p_i^k$, $V_S^{k+1}$ at all stages $k < k''$. For a given discount factor of the seller, if $k''$ is large enough there is necessarily a stage $k' < k''$, where the seller prefers to sell the good rather than wait. This is so because waiting for a long period is too costly to the seller, and if there was a long period of waiting to come the seller would be ready to offer the object even at a low price. This would be accepted by any buyer. Hence, at some stage $k' < k''$, there is a potential buyer. For a generic situation, this buyer is unique. Observe next that, as the possibility that the good be sold to another buyer gets further and further away in time, the maximum price that buyer $i$ is ready to accept gets closer and closer to his valuation $\pi_i$. This implies that if the set of potential buyers remains a singleton, $I^k = \{i\}$, for a long time after some stage, the maximum price that buyer $i$ is ready to pay at that stage gets close to $\pi_i$. When externalities are positive, this situation is clearly good to the seller, since it cancels the (bad) effect of positive externalities. One can show that there exist necessarily a buyer $i$, and a stage $k^* < k''$, such that for all $k < k^*$, $I^k = \{i\}$. Since $k''$ and hence $k^*$ can be chosen arbitrary large, we then infer that there cannot be infinitely many stages of delay in equilibrium. Hence there are at most finitely many stages of delay. Assume now (still by contradiction) that there is at least one stage of delay. Consider the largest (finite) $k''$ such that at stage $k''$ there is a delay. One can show that next to stage $k''$, there are many stages where the set of potential buyers is a singleton. By an argument similar to the one displayed above, the set of potential buyers at stage $k''$ is also this singleton, hence it is not empty. This shows that there is no delay in equilibrium.

All equilibria in Proposition 4.1 have a simple stationary structure: At every period, there is only one buyer to whom the seller makes acceptable offers at a price which is equal to that buyer's valuation.[13] Note that the discount factors considered in Proposition 4.1 may require that the seller be more patient than the buyers. However, the stationary equilibria just considered remain SPNE of the game $\Gamma_{\delta_S,\delta_B}$ even when the seller is no more patient than the buyers provided $\delta_B$, $\delta_S$ are close to one (see Corollary 4.4 below). Therefore delays are unlikely in the infinite-horizon game if externalities are positive and $\delta_B$, $\delta_S$ are close to one.

13. In the model where buyers make the offer with probability half, the structure of equilibria may be more complex (e.g. non-stationary). It is still true though that, under the conditions of Proposition 4.1, no SPNE displays delays.

### 4.2. *Cyclical delay with negative externalities*

Since the delay with negative externalities identified in the finite-horizon framework results from the fact that negative externalities can play the role of positive externalities, and since with positive externalities, we have just seen that delay is unlikely in the infinite-horizon framework, one might conjecture that delay disappears with negative externalities in the infinite-horizon game. We find that this intuition is not correct. The basic delay of the finite-horizon game may change into cyclical delay.

For the parameter values of Example 3.1, we establish that *all* equilibria have the feature that long periods of waiting alternate with short periods of activity: This is referred to as "cyclical delay". It will be clear that the cyclical delay phenomenon applies to any similar situation where some negative externalities play the role of positive externalities, and where the buyer who exerts the worst (negative) external effect has the highest valuation (see the comments after Result 4.2).

**Result 4.2** (Continuation of Example 3.1). *For all generic parameter values lying in a neighbourhood of those displayed in Example* 3.1, *there exists a discount factor* $\delta_B^* \in (0, 1)$ *and a function* $\delta_S^*(\cdot): (0, 1) \to (0, 1)$ *such that for all* $\sigma_{\delta_S, \delta_B} \in \mathscr{E}(\Gamma_{\delta_S, \delta_B})$ *with* $\delta_B \geq \delta_B^*$, $\delta_S \geq \delta_S^*(\delta_B)$, *and for all K, there exists* $k > K$, s.t. $I^k = \varnothing$. *Moreover, for all* $\sigma_{\delta_S, \delta_B} \in \mathscr{E}(\Gamma_{\delta_S, \delta_B})$, *there are infinitely many periods of activity which are separated by periods of waiting. Periods of activity are always of the form*: $I^{s+1} = \{1, 2\}$, $I^{s+2} = \{1, 2, 3\}$, $I^{s+3} = \{3\}$, *where* $I^s = I^{s+4} = \varnothing$.

*Proof.* See Appendix C. ||

*Remark.* Note that Result 4.2 may require that the seller be more patient than the buyers, but even if we consider the case where the seller and the buyers are equally (sufficiently) patient, it can be shown that all equilibria have infinitely many stages of delay (see footnote 14). Nevertheless, the period of activity could be more complex.

We now outline the structure of the proof. For any discount factors $\delta_S$, $\delta_B$, consider a SPNE in pure strategies with bounded recall $\sigma_{\delta_S, \delta_B} \in \mathscr{E}(\Gamma_{\delta_S, \delta_B})$. If $\delta_S$, $\delta_B$ are close to one, there are infinitely many periods of delay when the seller is more patient than the buyers. The intuition runs as follows. Consider an accumulation point, $(p_i, V_S^{\oplus 1})$, of the equilibrium values of $(p_i^k, V_S^{k+1})$, as $k$ varies. Assuming that at stage $K$, the values of $(p_i^K, V_S^{K+1})$ are those of the accumulation point $(p_i, V_S^{\oplus 1})$, we can construct the equilibrium values of the set of potential buyers at all stages preceding $K$. For discount factors sufficiently close to one, this construction leads either to the delay configuration or to the well-defined buyer configuration (see the discussion at the end of Section 3). If the delay configuration prevails, we conclude immediately that there are infinitely many periods of delay. If the well-defined buyer configuration prevails, we have to consider successively the cases where buyer 1, 2 or 3 is the well-defined buyer. Buyer 3 cannot be the well-defined buyer, since in such a case buyers 1 and 2 would be willing to pay more than buyer 3 in order to avoid buyer 3 getting the good. If buyer $i$ ($i = 1$ or 2) is the well-defined buyer, then for $k$ large enough, the price that buyer $i$ is willing to pay at stage $K-k$ decreases as $k$ increases because a potential sale to buyer 3 is further and further away in time. This price eventually converges to buyer $i$'s valuation which is close to $\pi$. When the seller is sufficiently patient, $\delta_S V_S^{K-k+1}$ decreases slower than $p_i^{K-k}$ as $k$ increases. Hence, there is $k$ such that all $p_j^{K-k}$ ($j = 1, 2, 3$) are less than $\delta_S V_S^{K-k+1}$: At stage $K-k$, there is

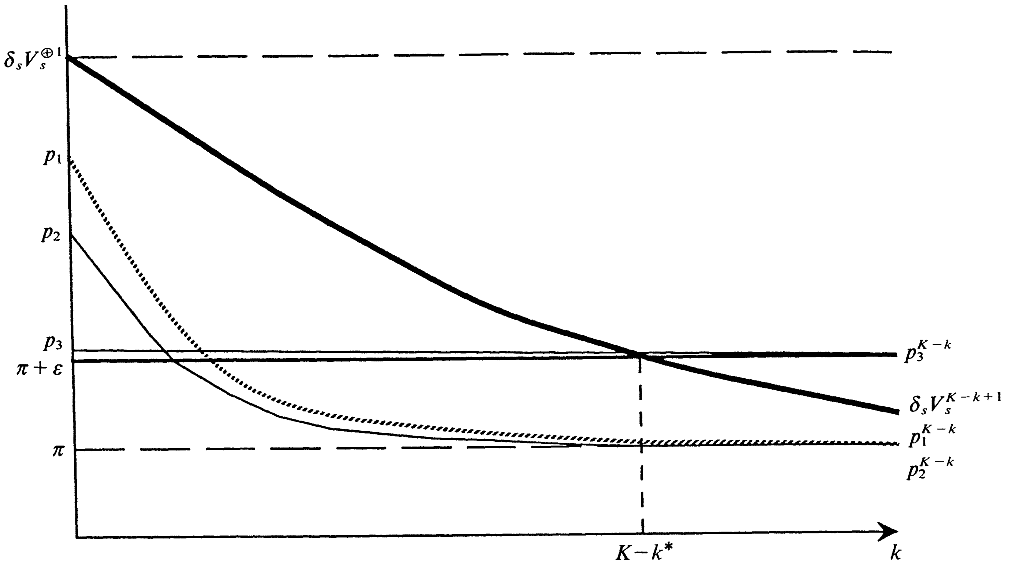



FIGURE 1
"Cyclical Delay"

a delay.[14] If the values of $(p_i^K, V_S^{K+1})$ are not exactly, but close to $(p_i, V_S^{\oplus 1})$, we also find that at stage $K-k$, there is a delay. Because the argument can be made for infinitely many such $K$ where $(p_i^K, V_S^{K+1})$ is close to $(p_i, V_S^{\oplus 1})$ (hence, infinitely many $K-k$), we find that there are always infinitely many periods of delay.

We next describe the structure of equilibria. We know that there are infinitely many periods of delay. Assume that for $K$ large, stage $K$ is a period of delay. The equilibrium values of $p_i^{K-k}$, $\delta_S V_S^{K-k+1}$ are constructed backwards (see Figure 1)[15].

Since the seller is more patient than the buyers, as $k$ increases, $p_i^{K-k}$ converges to $\pi_i$, quicker than $\delta_S V_S^{K-k+1}$ converges to zero. Since buyer 3 has the highest valuation, there exists $k^*$ such that $I^{K-k^*}=\{3\}$, all $p_i^{K-k^*}$ are close to $\pi_i$ and $\delta_S V_S^{K-k^*+1}$ is close to $\text{Max}_i(\pi_i)\approx\pi+\varepsilon$. By continuing the backward construction, we can conclude that for all equilibria with discount factors as in Result 4.2, there exist $K'$, $K''$, $K'''$ such that $I^{s+1}=\{1,2\}$, $I^{s+2}=\{1,2,3\}$, $I^{s+3}=\{3\}$, for $s=K', K'', K'''$, and $\forall k\in[K'+4, K'']\cup[K''+4, K''']$, $I^k=\varnothing$. Moreover, at stages $s+3$, $s=K', K'', K'''$, the maximum price that buyer $i$ is ready to accept is arbitrarily close to $\pi_i$, and the payoff to the seller is arbitrarily close to $\pi_3\approx\pi+\varepsilon$. Since the equilibrium values of $p_i^s$, $\delta_s V_S^{s+1}$ are very similar at stages $s=K', K''$, $K'''$, the set of potential buyers repeats itself in a cyclical fashion. This is why we refer to the delay in such equilibria as "cyclical delay".

What is the intuition for cyclical delay equilibria? In particular, why is there a period of waiting before stage $K''$? It is because of the effect occurring during the period of activity at stages $K''+1$, $K''+2$, $K''+3$. This effect is very similar to the deadline effect identified

14. When the buyers are as patient as the seller is, we also find that there is a stage with delay. That is because: Either (1) there is a $k$ such that all $p_j^{K-k}$ $(j=1,2,3)$ are less than $\delta_S V_S^{K-k+1}$, and at stage $K-k$, there is a delay; Or (2) at some stage $K-k$, the set of potential buyers is $I^{K-k}=\{i,3\}$ and buyer $i$ is ready to pay a price close to buyer 3's valuation, $\pi+\varepsilon$. Prior to stage $K-k$, the sets of potential buyers have a structure very similar to that occurring at stages $T-1$, $T-2$, $T-3$ in Example 3.1. Therefore at stage $K-k-3$, there is a delay.

15. Figure 1 is displayed for externality terms lying in a neighbourhood of those of Example 3.1.

in the finite-horizon framework. Now, why is there such a deadline effect? It is because the period of waiting that comes after stage $K''+3$ (and prior to stage $K'''$) forces the only potential buyer at stage $K''+3$ to be the buyer with highest valuation. Since this buyer is the buyer who exerts the (worst) negative externality, it reactivates a deadline effect of the finite-horizon type. Consequently, the cycle of periods of delay alternating with periods of activity (deadline effect) is self-enforcing. It is as if a deadline reappeared endogenously and cyclically.

Several comments are in order: First, in any equilibrium the length of the periods of waiting goes to infinity as the discount factors tend to one. Hence, the delay that separates two periods of activity may be arbitrarily long. Since in the limit of no discounting, delay is not costly to the seller, the number of stages may not be the proper measure to evaluate the length of delays. Alternatively, the length of delays in the (almost) cyclical equilibrium described above can be measured by the loss of payoff to the seller between two stages that separate a period of waiting, say stages $K'+3$ and $K''+1$. It is readily verified (see Appendix C) that this loss is arbitrarily close to $(10\alpha-24\varepsilon)/27$ as the discount factors get close to one. Hence as $\alpha$ grows, the loss may become large, which implies that, whatever the measure, the length of delays may be large.

The second comment concerns the presence of delay on the equilibrium path. The specific time in the (almost) cycle at which the equilibrium starts may be chosen arbitrarily. If the first stage of the game is a period of waiting, then the game will start with a period of delay. If the game starts at a period of activity where $I^1=\{1,2,3\}$, then the game ends immediately. The latter observation would not be valid if, in addition to the three buyers of Example 3.1, we considered a fourth *irrelevant* buyer (in the sense that he would never be a potential buyer).[16] For such a situation the probability that a period of delay be reached would always be strictly positive. In all equilibria, a long period of delay would be observed on the equilibrium path with positive probability.

In an infinite-horizon setting with complete and perfect information, Haller and Holden (1990) and Fernandez and Glazer (1991) have found equilibria with delayed agreement. We wish to emphasize that the cyclical delay phenomenon is very different from the delay established by those authors. Indeed, their equilibria use threats based on different stationary equilibria in different subgames (e.g. for different deviations). These stationary equilibria are chosen so that they force the players to choose a delayed agreement. It is readily verified that such equilibria require that the players have an unbounded memory capacity (at stage $t'>t+M$, in the subgame following a deviation at stage $t$, the behaviour strategies of the players have to keep track of that deviation).

### 4.3. *Stationary equilibria and complexity*

For the parameter values of Example 3.1, Result 4.2 shows that, there is no stationary equilibrium in pure strategies. This contrasts with most of the existing literature, where stationary games are in general shown to possess equilibria in pure stationary strategies. (Exceptions are Gurvich (1986) and Hendon and Tranaes (1991).) We now obtain necessary and sufficient conditions for the existence of equilibria in pure stationary strategies.

**Proposition 4.3.** *For generic situations, a* SPNE *in pure stationary strategies of* $\Gamma_{\delta_S,\delta_B}$ *exists for all values of* $\delta_S$, $\delta_B$ *sufficiently close to one if and only if there exists at least one buyer i such that* $\forall j\neq i$, $\pi_i>\pi_j+\alpha_{ij}$.

16. This can be guaranteed, for example, if his valuation as well as his external effects on the others are null.

*Proof.* See Appendix D. ||

If for all $i$ there exists $j$ such that $\pi_i < \pi_j + \alpha_{ij}$, Proposition 4.3 shows that there is no equilibrium in pure stationary strategies. The intuition is roughly as follows. Assume, by contradiction, that there exists a stationary equilibrium in pure strategies. We show (see Lemma D.1 in Appendix D) that there can be only one buyer to whom the seller makes acceptable offers. Let $i$ be this buyer. On the one hand, if $i$ rejects the offer, then he is not afraid that some other buyer will get the good (because of stationarity). Hence, he is only prepared to pay his valuation $\pi_i$. On the other hand, any buyer $j$, $j \neq i$ is sure that buyer $i$ will get the object in case $j$ himself does not acquire the good. If the discount factors are close to one, buyer $j$ is prepared to pay a price close to $\pi_j + \alpha_{ij}$. If there exists a buyer $j$ such that $\pi_j + \alpha_{ij} > \pi_i$, selling to $i$ only cannot be optimal for the seller, and the non-existence result follows.

We next apply Proposition 4.3 to the case of positive externalities:

**Corollary 4.4.** *Assume that all externalities are positive. The game* $\Gamma_{\delta_S, \delta_B}$ *has a* SPNE *in pure stationary strategies for all values of* $\delta_B$, $\delta_S$ *sufficiently close to one.*[17]

*Proof.* Let $i^* = \text{Arg max}_i\, \pi_i$. With positive externalities, $\forall j$, $\pi_{i^*} > \pi_j + \alpha_{i^*j}$ (since $\forall j$, $\alpha_{i^*j} < 0$), and we can apply Proposition 4.3. ||

We have seen in Result 4.2 that, when the seller is sufficiently patient relative to the buyers, all SPNE with bounded recall have a cyclical structure. Given such discount factors $\delta_B$, $\delta_S$, one may construct a SPNE that is exactly cyclical in the sense that the behaviour strategies (and not only the sequence $I^k$) have a cyclical structure. The length of the cycle is, as before, the number of stages between two periods of activity. Observe that, for such cyclical equilibria, the players need no longer identify the time period but only the point of the cycle at which they currently are, which is less demanding. It is also readily verified that the complexity of such equilibria is the length of the cycle, which is finite for fixed $\delta_B$, $\delta_S$: Such cyclical equilibria are the SPNE in pure strategies with bounded recall and minimum complexity. Note that, as the discount factors tend to one, the complexity of these cyclical equilibria tend to infinity. The latter observation is to be contrasted with Hendon and Tranaes (1991) who consider equilibria of complexity 2, but with unbounded memory capacity.

## 5. CONCLUDING COMMENTS

We have shown that transactions could be significantly delayed in situations involving externalities between buyers. Such delays may arise with negative externalities even when there is no imposed deadline.

We now present a final interpretation of our results. Ideally, the seller would like to threaten to sell the good to the most frightening buyer in case the selected effective buyer rejects her proposal. However, this happens to be non-credible for the seller, because in case of rejection the seller still faces the same problem and therefore is willing to sell to the same buyer. Given the credibility issues on the side of the seller, delay appears in a simple example to be the only way for the seller to credibly activate the threat. Of course,

17. There may be several such equilibria if there are several $i$ that satisfy: $\forall j$, $\pi_i > \pi_j + \alpha_{ij}$.

delay is costly to the seller, and the length of the delay measures, in some sense, how costly it is for her to make such a threat credible.

## APPENDIX A

*Proof of Lemma* 4.1. Consider stage $k$. The last $M$ matchings and actions are denoted by $h_M$. If $S$ and $i$ meet at stage $k$, the maximum price that $i$ is ready to accept in equilibrium is $p_i^k(h_M)$ and the minimum price that the seller is prepared to offer is $p_{S,i}^k(h_M)$. We first observe that the payoffs to the seller and the buyers can be computed at all stages $t$, $t>k$ using the truncated history at stage $k$, $h_M$, and the functions $p_i^t(\cdot)$, $p_{S,i}^t(\cdot)$. This is so because at all stages $t$, $t>k$, the strategies cannot depend on history prior to $h_M$. (For the computations, we use recursive formulae analogous to those defined in conditions (3.1) to (3.6) where $p_i^k(h_M)\geqq p_{S,i}^k(h_M)\Leftrightarrow i\in I^k(h_M)$, and $I^k(h_M)$ denotes the set of potential buyers at stage $k$ given $h_M$. We also denote by $V_i^{k+1}(h_M,p_i)$ and $V_S^{k+1}(h_M,p_i)$ the payoffs $i$ and $S$ obtain respectively at stage $k+1$ after the truncated history $(h_M,p_i)$ where $S$ and $i$ have met a stage $k$ and $S$ proposed $p_i$.)

The minimum price that the seller offers if he meets $i$ at stage $k$ (given $h_M$) is the payoff the seller can obtain by waiting (making an offer that is rejected). The offer $p_i$ is rejected if buyer $i$ can get more by waiting: $\delta_B V_i^{k+1}(h_M,p_i)>\pi_i-p_i$. Let $D_i=\{p_i/\delta_B V_i^{k+1}(h_M,p_i)>\pi_i-p_i\}$. $D_i$ is non-empty because the maximum price $P$ belongs to it. We have: $p_{S,i}^k(h_M)=\mathrm{Sup}_{p_i\in D_i}\,\delta_S V_S^{k+1}(h_M,p_i)$. Consider now stage $k+1$. By assumption, the payoffs to the seller and buyer $i$ at that stage can only depend on the last $M$ actions. The (truncated) history at stage $k+1$ is $(h_{M-1},p_i)$, where $h_{M-1}$ represents the last $M-1$ actions at stage $k$, and $p_i$ the offer at stage $k$. Hence, $V_S^{k+1}(h_M,p_i)$ is a function of $(h_{M-1},p_i)$ only, and the minimum price the seller can offer if he meets $i$ at stage $k$ can only depend on the last $M-1$ actions $h_{M-1}$.

Similarly, the maximum price that buyer $i$ can accept at stage $k$ if $S$ and $i$ meet is $p_i^k(h_M)=\mathrm{Arg\,max}_{p_i}\,(p_i/\pi_i-p_i\geqq\delta_B V_i^{k+1}(h_M,p_i))$. Since, by assumption, the payoff to buyer $i$ at stage $k+1$ can only depend on the last $M$ actions, we find that the maximum price $i$ can accept at stage $k$ can only depend on the last $M-1$ actions $h_{M-1}$.

By repeating inductively the above argument at stages $k+1$, $k+2,\ldots,$ $k+M$, we can conclude that the equilibria with bounded memory are history-independent. ||

## APPENDIX B

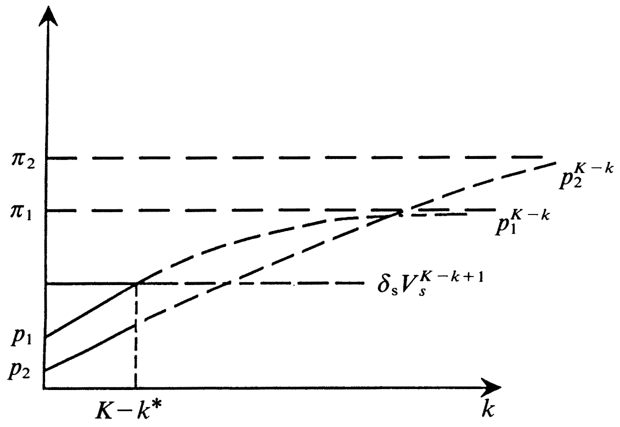
*Proof of Proposition* 4.1. Assume, by contradiction, that $\forall n$, $\exists\delta_B>1-(1/n)$, $\forall m$, $\exists\delta_S>1-(1/m)$ such that $\exists\sigma_{\delta_S,\delta_B}\in\mathscr{E}(\Gamma_{\delta_S,\delta_B})$ with $I^k=\varnothing$ for some $k$. We denote by $\delta_B^n$, $\delta_S^{n,m}$ any discount factors of the buyers and the seller, respectively, that satisfy the latter requirement. The associated equilibrium is denoted by $\sigma^{n,m}$. For given $n$, $m$ and $\sigma^{n,m}$, the maximum price that buyer $i$ is ready to accept at stage $k$ is denoted by $p_i^k(n,m)$, and the expected payoff to the seller at stage $k+1$ is denoted by $V_S^{k+1}(n,m)$. The sequence $(p_i^k(n,m),\ i=1,2,3;$ $V_S^{k+1}(n,m))_{k=1,\ldots\infty}$ has an accumulation point[18] that we denote by $(p_i(n,m),i=1,2,3;\ V_S^{\oplus 1}(n,m))$. For each $n$, we next consider an accumulation point of the sequence $(p_i(n,m),i=1,2,3;\ V_S^{\oplus 1}(n,m))_m$ that we denote by $(p_i(n),i=1,2,3;\ V_S^{\oplus 1}(n))$. As $n$ varies, this last sequence has an accumulation point denoted by $(p_i,i=1,2,3;\ V_S^{\oplus 1})$.

Consider now the situation where the discount factors of the seller and the buyers are exactly one, and the maximum price that each buyer $i$ is ready to accept at stage $K$ is $p_i^K=p_i$, while the payoff to the seller at stage $K+1$ is $V_S^{K+1}=V_S^{\oplus 1}$. For every stage $k<K$, we can backwards construct the set of potential buyers $I^k$ using conditions (3.1) to (3.6). We consider an arbitrary large $K$. There are two possibilities: Either the construction leads to the delay phenomenon, or it leads to the well-defined buyer case.

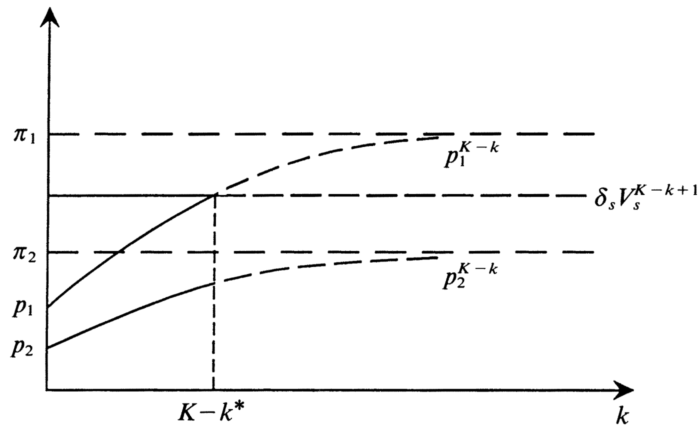
Assume first that the construction leads to the well-defined buyer case. For the backward construction associated with $\delta_B^n$, $\delta_S^{n,m}$, for $n$, $m$ sufficiently large, it must also be true that prior to some stage $K'\leqq K$, the set of potential buyers consists of the well-defined buyer only. (This is so because the maximum price that this buyer is ready to offer tends to his valuation as the probability that the good is sold to another buyer goes to zero. With positive externalities, this limit price is favourable to the seller.) Hence, there is no delay prior to $K'$. Since the reasoning can be made for arbitrarily large $K$ (and thus $K'$), $\sigma^{n,m}$ displays no delay. This is a contradiction to the assumption that there are periods with delay.

Assume next that the construction leads to the delay phenomenon. Then, for $n$ and $m$ large enough, we will show that prior to stage $K$, there is a stage $K'$ such that for all stages $k<K'$, there is no delay associated with $\sigma^{n,m}$. Since $K$ and $K'$ can be chosen arbitrarily large, $\sigma^{n,m}$ displays no delay and we have again a contradiction.

18. Such an accumulation point exists because the values are necessarily contained in a compact set.




(a)




(b)

FIGURE 2
No delay with positive externalities

To see how stage $K'$ can be constructed, we consider the case of two buyers first. Without loss of generality, stage $K$ is assumed to be a stage of delay, i.e. $V_S^{\oplus 1} > p_1$, and $V_S^{\oplus 1} > p_2$; and either $V_S^{\oplus 1} < \pi_1$, and $V_S^{\oplus 1} < \pi_2$ (case $a$), or $\pi_1 > V_S^{\oplus 1} > \pi_2$ (case $b$). We next proceed to the backward construction associated with $\delta_B^n$, $\delta_S^{n,m}$ for $n$, $m$ sufficiently large. (See Figures 2(a) and (b).)

In both cases $a$ and $b$, consider the first $k$ such that $p_i^{K-k} > \delta_S V_S^{K-k+1} > p_{-i}^{K-k}$ for $i=1$ or 2 (where $-i$ stands for the buyer other than $i$) and $\delta_S V_S^{K-k+2} > p_i^{K-k+1}$ for $i=1$ and 2. At stage $K-k+1$, the payoff to the seller is arbitrarily close to $V_S^{\oplus 1}$ if the seller is sufficiently patient relative to the buyers. Define $K' = K-k$. In case $a$, we have: $\pi_1 + \alpha_{21} \leqq p_1 < V_S^{\oplus 1} < \pi_2$ and $\pi_2 + \alpha_{12} \leqq p_2 < V_S^{\oplus 1} < \pi_1$. In case $b$, we have: $\pi_2 + \alpha_{12} < V_S^{\oplus 1} < \pi_1$. We can conclude that in both cases, prior to stage $K'$, the set of potential buyers is composed of $i$ only for $n$, $m$ large enough. For more than two buyers, consider the maximum accumulation point—still denoted $V_S^{\oplus 1}$—of the sequence $V_S^k$ where stage $k$ is a stage of delay: One can show that the reasoning for two buyers applies. (More precisely, as long as the reasoning for two buyers does not apply, one can find a strictly larger value of $V_S$ such that $(p_i,\ i=1,2,3;\ V_S)$ is also an accumulation point that corresponds to a stage of delay. Hence, when $V_S^{\oplus 1}$ is maximum, the argument must work.) ||

## APPENDIX C

*Proof of Result* 4.2. Assume, by contradiction, that $\forall n$, $\exists \delta_B > 1-(1/n)$, $\forall m$, $\exists \delta_S > 1-(1/m)$, such that $\exists \sigma_{\delta_S,\delta_B} \in \mathscr{E}(\Gamma_{\delta_S,\delta_B})$ $\exists K$, $\forall k > K$, $I^k \neq \varnothing$. That is, there are at most finitely many periods of delay associated with

$\sigma_{\delta_S,\delta_B}$. We denote by $\delta_B^n$, $\delta_S^{n,m}$ any discount factors of the buyer and the seller, respectively, that satisfy the latter requirement. Similarly, the associated equilibrium is denoted by $\sigma^{n,m}$. For given $n$, $m$, and $\sigma^{n,m}$, the maximum price that buyer $i$ is ready to accept at stage $k$ is denoted by $p_i^k(n, m)$. Similarly, the expected payoff to the seller at stage $k+1$ is denoted by $V_S^{k+1}(n, m)$. The sequence $p_i^k(n, m)$, $i=1, 2, 3$; $V_S^{k+1}(n, m))_{k=1,\ldots,\infty}$ has an accumulation point that we denote by $(p_i(n, m)$, $i=1, 2, 3$; $V_S^{\oplus 1}(n, m))$. For each $n$, we next consider an accumulation point of the sequence $(p_i(n, m)$, $i=1, 2, 3$; $V_S^{\oplus 1}(n, m))_m$ that we denote by $(p_i(n)$, $i=1, 2, 3$; $V_S^{\oplus 1}(n))$. As $n$ varies, this last sequence has an accumulation point denoted by $(p_i, i=1, 2, 3: V_S^{\oplus 1})$.

Consider now the situation where the discount factors of the seller and the buyers are exactly one, and the maximum price that each buyer $i$ is ready to accept at stage $K$ is $p_i^K = p_i$, while the payoff to the seller at stage $K+1$ is $V_S^{K+1} = V_S^{\oplus 1}$. For every stage $k<K$, we backwards construct the set of potential buyers $I^k$ using conditions (3.1) to (3.6). We consider an arbitrary large $K$. We know that there are two possibilities. Either the construction leads to the delay phenomenon, or it leads to the well defined buyer case.

Assume first that the construction leads to the delay phenomenon. Then we get a contradiction to the assumption that there are at most finitely many periods of delay associated with $\sigma^{n,m}$, for $n$ and $m$ large enough. (This is so because, as $n$ and $m$ get large, $(p_i, i=1, 2, 3; V_S^{\oplus 1})$ is approximated infinitely many times in the SPNE $\sigma^{n,m}$. For any finite number of steps, one can guarantee that the backward construction associated with $\delta_B^n$, $\delta_S^{n,m}$ yields the same sets of potential buyers as those when there is no discounting.)

Hence, we can assume that the backward construction leads to the well-defined buyer case. We review successively the cases where 3, 1 or 2 is the well defined buyer. Assume that the well-defined buyer is 3: This implies that there are stages where buyers 1 and 2 will be ready to pay approximately $\pi+\alpha$ for the object. The maximal price that buyer 3 is ready to offer is necessarily close to $\pi+\varepsilon$ (because buyer 3 is not really concerned with who gets the good), and this is less than $\pi+\alpha$. Hence, 3 cannot be the well-defined buyer.

Thus, we can assume that either buyer 1, or 2, is the well-defined buyer. Without loss of generality, we assume that 1 is the well-defined buyer. It must be that at some further stage $k$, buyer 3 is a potential buyer, i.e. $3 \in I^k$. Otherwise, the current maximum price that 1 is ready to accept would be close to $\pi$, which is less than the maximum price that 3 is ready to accept approximately $(\pi+\varepsilon)$, a contradiction to the premise that 1 is the well-defined buyer. Moreover, if buyer 1 is the only potential buyer for a long time, the maximum price that buyer 2 is ready to accept is arbitrarily close to $\pi$, and the payoff to the seller is arbitrarily close to the maximum price that buyer 1 is ready to accept. This implies that, for the above accumulation point, we may assume that $V_S^{\oplus 1}$ is close to $p_1$ and strictly greater than $\pi+\varepsilon$,[19] while $p_2$, $p_3$ are close to $\pi$ and $\pi+\varepsilon$, respectively. Consider now any discount factors $\delta_B$, $\delta_S<1$. Assuming that $p_i^K=p_i$ and $V_S^{K+1}=V_S^{\oplus 1}$, we can proceed to the backward construction for every stage $K-k$, $k>0$. Because buyer 1 discounts the future $p_1^{K-k}$ will necessarily decrease to buyer 1's valuation, i.e. almost $\pi$, as $k$ increases, whereas if $S$ is sufficiently patient, $V_S^{K-k}$ will remain close to $V_S^{\oplus 1}$. For such discount factors, there exists a finite $k$ such that $\delta_S V_S^{K-k+1} > p_i^{K-k}$ for $i=1, 2, 3$, which results in $I^{K-k}=\varnothing$. We now obtain a contradiction to the assumption that there are at most finitely many periods of delay associated with $\sigma^{n,m}$. Indeed, fix $n$ large. By considering $m$ sufficiently large, we can guarantee that $\delta_S^{n,m}$ is arbitrarily close to one. By considering infinitely many (large) $K$, the above analysis shows that $\sigma^{n,m}$ displays infinitely many stages of delay.

We have shown that there are infinitely many stages of delay in any equilibrium. We now establish that periods of activity are composed of three stages only. The idea is to consider a stage $K$ with delay, where $K$ is assumed to be large enough. For a more patient seller, the backward construction leads to buyers' maximum prices and seller's expected payoff which are as depicted in Figure 1. As $k$ increases, the maximum price buyer $i$ is ready to accept at stage $K-k$ will decrease to his valuation $\pi_i$, whereas the expected payoff to the seller at stage $K-k+1$ can be chosen to be arbitrarily close to his payoff at stage $K+1$ if the seller is sufficiently patient. As $k$ becomes even larger, the expected payoff of the seller at stage $K-k+1$ will eventually decrease, and for some $k^*$, i.e. at some stage $s+3=K-k^*$, buyer 3 will be the only potential buyer at a price close to his valuation, since buyer 3 has the highest valuation (i.e. $I^{s+3}=\{3\}$). Using that all $p_i^{s+3}$ are close to $\pi_i$ and $V_S^{s+4}$ is close to $\pi+\varepsilon$, we obtain that $I^s=\varnothing$, $I^{s+1}=\{1,2\}$, $I^{s+2}=\{1,2,3\}$ as required $(p_1^s \approx p_2^s \approx \pi+(8\alpha/27) < V_S^{s+1} \approx \pi+(1/27)(10\alpha+3\varepsilon)$. Therefore the payoff to the seller oscillates between $\pi+\varepsilon$ and $\pi+(1/27)(10\alpha+3\varepsilon)$, the difference of which is $(1/27)(10\alpha-24\varepsilon)$.) ||

## APPENDIX D

The analysis of stationary equilibria in pure strategies relies on the following Lemma which is related to the result in Jehiel and Moldovanu (1995) stating that equilibria of the finite-horizon game, if not of the delay type,

19. Since buyer 3 is an effective buyer at least once in the future, buyer 1 is ready to pay at least $\pi+\alpha/3$ just before, which shows that the seller can secure strictly more than $\pi+\varepsilon$, for $\alpha>3\varepsilon$.

are necessarily of the well-defined buyer type:

**Lemma D.1.** *For a generic situation, assume that $\Gamma_{\delta_S,\delta_B}$ possesses stationary equilibria in pure strategies, and let $\sigma$ be one of them. Then if $\delta_S$, $\delta_B$ are sufficiently close to one, the set of potential buyers associated with $\sigma$ is a singleton.*

*Proof.* Let $I$ denote the set of potential buyers associated with the stationary equilibrium $\sigma$: $i \in I$ if and only if whenever $S$ meets $i$, $S$ proposes a price $p_i$ (which is constant by stationarity) and $i$ accepts. $I$ is non-empty because of the stationarity. Denote by $C$ the cardinality of $I$. Denote by $V_i$ the payoff to $i$ when $\sigma$ is played. Assume, by contradiction, that $C>1$. If $i \in I$, we obtain:

$$V_i = \frac{1}{N}(\pi_i - p_i) + \frac{1}{N}\sum_{j\in I, j\neq i}(-\alpha_{ji}) + \frac{N-C}{N}\delta_B V_i. \tag{D.1}$$

The price $p_i$ must satisfy:

$$\pi_i - p_i = \delta_B V_i. \tag{D.2}$$

From (D.1) and (D.2), we obtain:

$$p_i = \pi_i + \frac{\sum_{j\in I, j\neq i}(-\alpha_{ji})}{(N/\delta_B) - N + C - 1} \underset{\delta_B \to 1}{\to} \pi_i + \frac{\sum_{j\in I, j\neq i}(-\alpha_{ji})}{C-1}. \tag{D.3}$$

Let $i^*$ be such that $i^* + \operatorname{Arg\,max}_{i\in I}(\pi_i + (\sum_{j\in I, j\neq i}(-\alpha_{ji}))/(C-1))$. For generic situations, $i^*$ is unique. Hence for $\delta_B$ close enough to one, we obtain that $\forall i \in I, i \neq i^*, p_i < p_{i^*}$. Now if $\delta_S$ is sufficiently close to one, the seller is always better off waiting for $i^*$ rather than selling the object at a price lower than $p_{i^*}$. This is a contradiction to $C>1$. ||

*Proof of Proposition* 4.3. Assume first that $\forall i$, $\exists j$ such that $\pi_i < \pi_j + \alpha_{ij}$. We show that $\Gamma_{\delta_S,\delta_B}$ has no SPNE in pure stationary strategies for $\delta_S$, $\delta_B$ close to one. Assume by contradiction that such a SPNE, $\sigma$, exists. We will obtain a contradiction to Lemma D.1. We know from Lemma D.1 (we use the same notation here) that for some $i$, $I=\{i\}$. For that $i$, we have:

$$V_i = \frac{1}{N}(\pi_i - p_i) + \frac{N-1}{N}\delta_B V_i \tag{D.4}$$

$$\pi_i - p_i = \delta_B V_i. \tag{D.5}$$

Equations (D.4) and (D.5) yield together:

$$p_i = \pi_i, \; V_i = 0. \tag{D.6}$$

Consider now buyer $j$, $j \neq i$. We have:

$$V_j = \frac{1}{N}(-\alpha_{ij}) + \frac{N-1}{N}\delta_B V_j. \tag{D.7}$$

If $S$ meets $j$, this buyer would be prepared to pay a price $p_j$ such that:

$$\pi_j - p_j = \delta_B V_j. \tag{D.8}$$

Equations (D.7), (D.8) yield together:

$$p_j = \pi_j + \frac{\alpha_{ij}}{(N/\delta_B) - (N-1)} \underset{\delta_B \to 1}{\to} \pi_j + \alpha_{ij}. \tag{D.9}$$

Since by assumption $\pi_j + \alpha_{ij} > \pi_i$, for $\delta_B$, $\delta_S$ sufficiently close to one $p_j > p_i$, and the seller is better off by making serious offers to $j$, which contradicts the fact $i$ is the only potential buyer.

For the converse part, let $i^*$ be such that $\forall j \neq i^*$, $\pi_{i^*} > \pi_j + \alpha_{i^*j}$. Consider the following strategy profile $\sigma$. The seller always offers $p_{i^*} = \pi_{i^*}$ to $i^*$, and waits whenever he meets other buyers. Buyer $i^*$ accepts only offers less than or equal to $\pi_{i^*}$. Buyer $j$, $j \neq i$, accepts only offers less than or equal to $\pi_j + (\alpha_{i^*j})/[(N/\delta_B) - (N-1)]$. The reader will have no difficulty proving that for $\delta_B$, $\delta_S$ sufficiently close to one, $\sigma$ is SPNE (in pure stationary strategies). ||

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