Efficient Sequential Assignment with Incomplete Information

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Abstract

We study the welfare maximizing assignment of several heterogenous, commonly ranked objects to impatient agents with privately known characteristics who arrive sequentially according to a Poisson or renewal process. We focus on two cases: 1. There is a deadline after which no more objects can be allocated; 2. The horizon is potentially infinite and there is time discounting. We first characterize all implementable allocation schemes and show that the dynamically efficient allocation falls in this class. We then obtain several properties of the welfare maximizing policy using stochastic dominance measures of increased variability and majorization arguments. These results yield upper/lower bounds on efficiency for large classes of distributions of agents' characteristics or of distributions of inter-arrival times for which explicit solutions cannot be obtained in closed form. We also propose redistribution mechanisms that 1) implement efficient allocation 2) satisfy individual rationality 3) never run a budget deficit 4) may run a budget surplus that vanishes asymptotically.

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1 Introduction

We study efficient, individual rational and budget-balanced schemes for the following dynamic mechanism design problem: the designer wants to assign a fixed, finite set of heterogenous objects to a sequence of randomly arriving agents with privately known characteristics. Monetary transfers are feasible. The objects are substitutes, and each agent derives utility from at most one object. Moreover, all agents share a common ranking over the available objects, and values for objects have a multiplicative structure involving the agents' types and objects' qualities. In one formulation we assume that there is a deadline by which all objects must be sold, in another we assume a discounted infinite horizon (in both scenarios, time is a continuous variable).

Examples of such settings include the dynamic allocation of limited resources among incoming projects, the allocation of limited research facilities among research units (e.g., telescope time), the assignment of dormitory rooms to potential tenants, and the allocation of available positions to arriving candidates. The yield-management literature has analyzed the simpler models of allocating identical objects (e.g., seats on an aeroplane, or hotel rooms) from the point of view of revenue-maximization[1]. Our model also shares several common features to the classical job search models[2]. The main difference is that in that literature it is usually assumed that the stream of the job offers is generated by a non-strategic player, without private information. Hence, implementation issues do not arise there. [3]

Compared to a static setting, the new trade-off is between an assignment today and the valuable option of assigning it in the future, possibly to an agent who values it more. Since the arrival process of agents is stochastic, the "future" on which the option value depends may never materialize (if there is a deadline) or it may be farther away in time, and thus discounted.

First, we characterize all dynamically implementable deterministic allocation policies.

[1] See McAfee and te-Velde [14], and Gershkov and Moldovanu [9] for several reference to that large literature

[2] For extensive surveys of the search literature see Lippman and McCall [13] , and Mortensen [15]

[3] Other differences are: the job search model corresponds here to the one object case, and sampling has an explicit cost.

Such policies are described by partitions of the set of possible agent types: an arriving agent gets the best available object if his type lies in the highest interval of the partition, the second best available object if his type lies in the second highest interval, and so on. These intervals may depend on the point in time of the arrival, and on the composition of the set of available objects at that point in time. For implementable allocation policies we derive the associated menus of prices (one menu for each point in time, and for each subset of remaining objects) that implement it, and show that these menus have an appealing recursive structure: each agent who is assigned an object has to pay the value he displaces in terms of the chosen allocation. It allows us to verify the implementability of the dynamically efficient allocation policy under Poisson arrivals, which has been characterized for the complete information case - via a system of differential equations - by Albright [1][4]. Since that policy is deterministic, Markovian and has the form of a partition, it can be implemented also in our private information framework by the dynamic price schedules identified above, which coincide then with a dynamic version of the Vickrey-Clarke-Groves mechanism. Dolan [7] used a dynamic version of the Vickrey-Clarke-Groves mechanism in order to achieve welfare maximization in queues with random arrivals and with incomplete information about the agents' characteristics. Dynamic extensions of VCG schemes (for much more general situations than those considered here) have recently attracted a lot of interest - see for example Athey and Segal [3], Bergemann and Välimäki [4], and Parkes and Singh [17]. Gershkov and Moldovanu[10] analyze the limitations encountered in a framework where the designer needs also to learn about the distribution of agents' characteristics.

A somewhat surprising feature is that the cutoff curves defining the intervals in the time-dependent partitions that characterize the dynamic welfare maximizing policy in our model depend only on the cardinality of the set of available objects, but not on the exact composition of that set. This is due here to the multiplicative structure of the agents' valuations for objects.

[4] Derman, Lieberman and Ross [6] introduced the basic assignment model in a framework with a finite number of periods (time is discrete), and one arrival per period. An early paper that uses optimal stopping theory to characterize the efficient assignment of a single object to randomly arriving agents in continuous time is Elfving [8].

The dynamically efficient allocation policy can be explicitly computed if the distribution of agents' types is exponential, while this is not often the case for general distributions[5]. But, we use comparative static results in order to bound the cutoff curves in the welfare maximizing policy (and the associated expected welfare) for the important and large, non-parametric classes of distributions that second-order stochastically dominate (are dominated by) the exponential distribution - these are the so called *new better (worse) than old in expectation* distributions. These bounds can be used for evaluation other simple, but not necessarily optimal allocation policies. We show that a decrease in the second order stochastic sense in the distribution of agents' types (which implies an increase in variability) leads to an increase in expected welfare in the dynamic assignment problem. The proof of this result uses several simple insights from *majorization theory.*[6] Majorization by a vector of weighted sums of order statistics plays also the main role in assessing the welfare loss due to the sequential nature of the allocation process versus the scenario where the allocation can be delayed till after all arrivals and types have been observed.

While the above comparative static result holds for both the deadline model and for the discounted, infinite horizon model, in the latter case - where the welfare maximizing policy can be characterized and turns out to be time-independent for general *renewal* arrival processes - we also examine the effect on expected welfare of a stochastic increase in the distribution of inter-arrival times in the sense of the *Laplace-transform order.* This stochastic order is much weaker than second order stochastic dominance.[7] In particular, more variability in inter-arrival times leads to higher expected welfare. For example, in the case of one object, bounds on expected welfare relative to an exponential distribution of agents' types and a Poisson arrival process can be expressed in terms of the well known

[5] In a model with identical objects, McAfee and te Velde [14] compute the dynamic welfare maximizing policy for a Pareto distributions of agents' values, and show that it coincides then with the revenue optimizing policy. For a general comparison of the welfare-maximizing and revenue-maximizing policies see Gershkov and Moldovanu [9]. See also these papers for other references to the relevant literature on yield/revenue management.

[6] See Hardy, Littlewood and Polya, [12].

[7] Any two distributions that are ranked in the second order stochastic dominance order are also ranked in the Laplace-transform order, but the opposite is not true.

Lambert-W function.

In general, it is not enough to focus on the physical allocation in order to ensure efficiency since one also has to describe what happens to the monetary payments. In some frameworks these accrue to a third-party (e.g., a seller in an auction) and physical efficiency is thus what matters. In other settings, the implementation of the efficient physical allocation generates a deficit or a surplus. If there is no third party, full efficiency calls then for budget-balancedness. It is well known that in the static setup no VCG mechanism can generally be both individual rational and budget balanced. Guo and Conitzer [11] and Moulin [16] analyze surplus minimizing VCG schemes in the static environment, and derive bounds on the ensuing efficiency loss.

For the study of efficient, individual rational and budget-balanced mechanisms in our frameworks, we distinguish between two scenarios: 1) Both physical assignments and monetary payments must take place upon the agents' arrival; 2) Payments can be postponed to later stages. While in the second scenario budget balancedness can be always reached, in the first scenario budget balancedness can only be reached asymptotically as the arrival rate (or the time up to the deadline) increases without bound.

The rest of the paper is organized is follows: In Section 2 we present the continuous-time model of sequential assignment of heterogenous objects to randomly arriving, privately informed agents. Section 3 focuses on a characterization of implementable policies, and of the associated menus of dynamic prices that implement such policies. In Section 4 we present a Theorem, due to Albright [1] that determines the dynamic welfare maximization policy in a framework with complete information and Poisson arrivals. In Subsection 4-1 we apply Albright's theorem to a setting where the allocation of all available objects must occur before a known deadline, and we consider the effect of changes in the distribution of agents' types. We also assess the welfare loss due to the sequential nature of the allocation process. Subsection 4-2 deals with an infinite horizon model with exponential discounting. There we consider both the effects of changes in the distribution of agents' types, and in the distribution of inter-arrival times. Section 5 focuses on budget balancedness and we look at mechanisms that redistribute the raised revenue from the allocation process. Section 6 concludes. Most of the proofs are relegated to an Appendix.

2 The Model

There are n items (or objects). Each item i is characterized by a "quality" q_i. Each agent j is characterized by a "type" x_j. Agents arrive according to a (possibly non-homogenous) Poisson process with intensity $\lambda(t)$, and each can only be served upon arrival (i.e., agents are impatient). After an item is assigned, it cannot be reallocated in the future. For some results we relax the Poisson assumption, and we allow for a more general renewal stochastic process to describe arrivals.

An agent with type x_j who obtains an item with characteristic q_i enjoys a utility of $q_i x_j$. If an item of quality q_i is assigned to an agent with type x_j at time t, then the utility for the designer is given by $r(t)q_i x_j$ where r is a piecewise continuous, non-negative, non-increasing discount function which satisfies $r(0) = 1$.

While the items' types $0 \leq q_n \leq q_{n-1} \leq ... \leq q_1$ are assumed to be known constants, the agents' types are assumed to be represented by independent and identically distributed random variables X_i on $[0, +\infty)$ with common c.d.f. F that has a support $[0, \Upsilon]$ with $\Upsilon \leq \infty$. The realization of X_i is private information of agent i. We assume that each X_i has a finite mean, denoted by μ, and a finite variance.

3 Implementable Policies

Without loss of generality, we restrict attention to direct mechanisms where every agent, upon arrival, reports his characteristic x_i and where the mechanism specifies an allocation (which item, if any, the agent gets) and a payment. As we shall see, the schemes we develop also have an obvious and immediate interpretation as indirect mechanisms, where the designer sets a time-dependent menu of prices, one for each item, and the arriving agents are free to choose out that menu.

An allocation policy is called *deterministic* and *Markovian* if, at any time t, and for any possible type of agent arriving at t, it uses a non-random allocation rule that only depends on the arrival time t, on the declared type of the arriving agent, and on the set of items available at t, denoted by Π_t. Thus, the policy depends on past decisions only via the state variable Π_t. We restrict attention to interim-individually rational policies, where no agent ever pays more than the utility obtained from the physical allocation.

Denote by $Q_t : [0, +\infty) \times \Pi_t \to \Pi_t \cup \emptyset$ a non-randomized Markovian allocation policy for time t and by $P_t : [0, +\infty) \times \Pi_t \to \mathbb{R}$ the associated payment rule. Denote also by k_t the cardinality of set Π_t.

The next Proposition shows that a non-randomized, Markovian allocation policy is implementable if and only if it is based on a partition of the agents' type space.[8] In other words, implementability reduces here to setting a menu of prices, one for which object, from which the arriving agent has to choose.

Proposition 1 *Assume that Π_t is the set of objects available at time t, and assume that $q_j \neq q_k$ for any $q_j, q_k \in \Pi_t$, $j \neq k$.*

1. *A non-randomized, Markovian policy Q_t is implementable if and only if there exist $k_t + 1$ functions $\infty = y_{0,\Pi_t}(t) \geq y_{1,\Pi_t}(t) \geq y_{2,\Pi_t}(t) \geq \cdots \geq y_{k_t,\Pi_t}(t) \geq 0$, such that $x \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t)) \Rightarrow Q_t(x, \Pi_t) = q_{(j)}$ where $q_{(j)}$ denotes the j'th highest element of the set Π_t, and such that $x < y_{k_t,\Pi_t}(t) \Rightarrow Q_t(x, \Pi_t) = \emptyset$.*[9]

2. *The associated payment scheme is given by $P_t(x, \Pi_t) = \sum_{i=j}^{k_t}(q_{(i)} - q_{(i+1)})y_{i,\Pi_t}(t) + S(t)$ if $x \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$ where $S(t)$ is some allocation- and type-independent function.*

Proof. $\Longrightarrow$ If two reports of the agent that arrives at t lead to the same physical allocation, then, in any incentive compatible mechanism, the associated payments should be the same as well. Denote by P_j the payment that will be charged for the object with quality q_j. A direct mechanism is equivalent to a mechanism where the agent arriving at time t chooses an object and a payment from a menu $(q_j, P_j)_{j=1}^{k_t}$. If some type x prefers the pair (q_k, P_k) over any other pair (q_l, P_l) with $q_k > q_l$, then any type $\widetilde{x} > x$ also prefers (q_k, P_k) over (q_l, P_l). This implies that $Q_t(\widetilde{x}, \Pi_t) \geq Q_t(x, \Pi_t)$ for any t and

[8] The result holds for any deterministic policy. But, since the rest of the analysis focuses on the Markov case, and in order to save on notational complexity, we consider only this case here.

[9] Types at the boundary between two intervals can be assigned to either one of the neighboring elements of the partition. That is, if $x_i \in \{y_{k_t,\Pi_t}(t), y_{k_t-1,\Pi_t}(t), ..., y_{2,\Pi_t}(t), y_{1,\Pi_t}(t)\}$, then $Q_t(y_{i,\Pi_t}(t), \Pi_t) \in \{q_i, q_{i+1}\}$, $i = 1, 2, .., k_t$.

Π_t. Finally, noting that $Q_t(x, \Pi_t) = \emptyset$ is equivalent to allocating an object with quality equal to zero, implies that an agent who arrives at time t gets object $q_{(k)}$ if he reports a type contained in the interval $(y_{k,\Pi_t}(t), y_{k-1,\Pi_t}(t))$. A similar argument shows that $Q_t(y_{i,\Pi_t}(t), \Pi_t) \in \{q_{(i+1)}, q_{(i)}\}$ for $i \in \{1, 2, ..., k_t\}$.

$\Longleftarrow$ The proof is constructive: given a partition-based policy, we design a payment scheme $P_t(x, \Pi_t)$ that, for any $j \in \{1, ..., k_t\}$, induces type $x \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$ to choose the object with type $q_{(j)}$. Without loss of generality, we assume that an agent whose type is on the boundary between two intervals in the partition chooses the item with higher type. Consider then the following payment scheme

$$P_t(x, \Pi_t) = \sum_{i=j}^{k_t} (q_{(i)} - q_{(i+1)}) y_{i,\Pi_t}(t) + S(t), \text{ if } \ x \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$$

where $S(t)$ is some allocation- and type-independent function. Note that type $x = y_{j,\Pi_t}(t)$ is indifferent between $(q_{(j)}, P_j)$ and $(q_{(j+1)}, P_{j+1})$. Moreover, any type above $y_{j,\Pi_t}(t)$ prefers $(q_{(j)}, P_j)$ over $(q_{(j+1)}, P_{j+1})$, while any type below, prefers $(q_{(j+1)}, P_{j+1})$ over $(q_{(j)}, P_j)$. Therefore, any type $x \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$ prefers $(q_{(j)}, P_j)$ over any other pairs in the menu.[10] ■

4 The Dynamically Efficient Policy

Albright [1] characterized the allocation policy that maximizes the total expected welfare from the designer's point of view in a complete-information model. That is, in his model, the type of the arriving agent becomes public information upon arrival. His main result is:

Theorem 1 *(Albright, [1]) There exist n unique functions* $y_n(t) \leq y_{n-1}(t) ... \leq y_1(t)$, $\forall t$ *, which do not depend on the q's such that:*

1. *If an agent with type x arrives at a time t, it is optimal to assign to that agent the j'th highest element of Π_t if $x \in [y_j(t),\ y_{j-1}(t))$,where $y_0 \equiv \infty$, and not to assign any object if $x < y_{k_t}(t)$.*

[10] If there are some identical objects, there exist implementable policies that do not take the form of partitions. But, for each such policy, there exists another implementable policy that is based on a partition, and that generates the same expected utility for all agents and for the designer.

2. *For each k, the function $y_k(t)$ satisfies :*

   (a) $\lim_{t\to\infty} r(t)y_k(t) = 0$

   (b) $\frac{d[r(t)y_k(t)]}{dt} = -\lambda(t)r(t)\int_{y_k}^{y_{k-1}}(1-F(x))dx \leq 0$.

3. *The expected welfare starting from time t is given by $\left[\sum_{i=1}^{k_t} r(t)q_{(i)}y_i(t)\right]$, where $q_{(i)}$ is the i'th highest element of Π_t.*

The surprising element in the above result is that the dynamic welfare maximizing cutoff curves $y_j(t)$ do not depend on the items' characteristics. In other words, although the Markov decision problem is one with $2^n - 1$ states, corresponding to all possible non-empty subsets of items Π_t, the welfare maximizing policy is such that, at each point in time t, and for each type of the arriving agent, the allocation decision is only contingent on the cardinality of Π_t, k_t. Moreover, since selling one object is equivalent to exchanging one of the currently available items with an item having a type equal to zero, the above observation implies that after any sale at time t, the $k_t - 1$ curves that determine the optimal allocation from time t on, coincide with the $k_t - 1$ highest curves that were relevant for the decision at time t. Thus, in effect, there are only n relevant states for the decision maker instead of $2^n - 1$.

To understand the intuition behind this result, assume for simplicity that at time t there are two objects $q_1 > q_2$ and that the relevant cutoffs are $y_1^e > y_2^e$. Consider the effect of a small shift in the highest cut-off from y_1^e to $y_1^e + \epsilon$. This shift has any effect only if an agent indeed arrives at t. Second, the shift has no effect if the arriving agent has a value above $y_1^e + \epsilon$ or below y_1^e. If, however, at time t an agent with value y_1^e arrives, then this shift switches the object he gets from q_1 to q_2 and therefore switches the object available for future allocation from q_2 to q_1. Therefore, the effect of the shift on the social welfare is

$$\begin{aligned} & f\left(y_1^e\right)\left(q_1 y_1^e + W\left(q_2, t\right) - q_2 y_1^e - W\left(q_1, t\right)\right) \\ = \ & \left(q_1 - q_2\right) f\left(y_1^e\right)\left(y_1^e - W\left(1, t\right)\right) \end{aligned}$$

where $W(q,t)$ denotes here the expected welfare at time t if only one object with type q remains, given that the optimal policy is followed from time t on, where the equality follows since $W(q,t)$ is linear in q. The above expression is linear and separable in the difference $(q_1 - q_2)$, and therefore the optimal cutoff - where the total effect of a shift should be equal to zero - will not depend on this difference.

Since the Markovian, deterministic policy described in Theorem 1 has the form of a partition, it can be implemented by the payments (or by prices in an indirect mechanism) described in Proposition 1. Note that Theorem 1 -3. implies that the payment

$$P_t(x, \Pi_t) = \sum_{i=j}^{k_t} (q_{(i)} - q_{(i+1)}) y_{i,\Pi_t}(t)$$

can be interpreted as the expected externality imposed on other agents by an agent that arrives at time t who gets the object with the j-th highest type among those remaining at time t. In other words, for the dynamically efficient policy, our implementing mechanism coincides with a dynamic Clarke-Groves-Vickrey mechanism, as studied by Athey and Segal [3], Bergemann and Välimäki [4], and Parkes and Singh [17].

4.1 The Dynamic Efficient Allocation with a Deadline

In this Section we apply Theorem 1 to a framework with deadline T after which all objects perish. Our main result shows that an increase in the variability of the distribution of the agents' values (while keeping a constant mean) increases expected welfare. We want to emphasize that this result holds even if all available objects are identical!

Besides its intrinsic interest, this result allows us to bound the welfare maximizing cut-off curves (and thus the expected welfare) for large and important families of distributions for which an explicit solution of the system of differential equations that characterizes the efficient policy is not available (see Theorem 1-2b).

We assume that the discount rate satisfies:

$$r(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le T \\ 0 & \text{if} \quad t > T \end{cases}$$

It is then obvious that the dynamically efficient policy needs to satisfy

$$y_1(T) = y_2(T) = ... = y_n(T) = 0$$

Example 1 *(Exponential distribution) Assume that the arrival process is homogenous with rate* $\lambda(t) = \lambda$ *normalized to be* 1, *and that the distribution of agents' types is exponential, i.e.,* $F(x) = 1 - e^{-x}$. *From Theorem 1, we obtain the following system of differential equations that characterize cut-off curves in the dynamic welfare maximizing policy:*

$$\begin{aligned} y_1' &= -\int_{y_1}^{\infty} e^{-x} dx = -e^{-y_1}; \\ y_i' &= -\int_{y_i}^{y_{i-1}} e^{-x} dx = e^{-y_{i-1}} - e^{-y_i}, \; i > 1 \end{aligned}$$

with initial conditions $y_i(T) = 0, \; i \geq 1$. *For example, the solution to this system for the case of three objects is:*

$$\begin{aligned} y_1(t) &= \ln(1 + T - t); \; y_2(t) = \ln\left(1 + \frac{(T-t)^2}{2(1+T-t)}\right) \\ y_3(t) &= \ln\left(1 + \frac{(T-t)^3}{3[(T-t)^2 + 2(1+T-t)]}\right) \end{aligned}$$

For the main results in this Section, we need a well- known concept, due to Hardy, Littlewood and Polya [12].

Definition 1 *For any* $n-$*tuple* $\gamma = (\gamma_1, \gamma_2, .., \gamma_n)$ *let* $\gamma_{(j)}$ *denote the* j*th largest coordinate (so that* $\gamma_{(n)} \leq \gamma_{(n-1)} \leq ... \leq \gamma_{(1)}$. *Let* $\alpha = (\alpha_1, \alpha_2, .., \alpha_n)$ *and* $\beta = (\beta_1, \beta_2, .., \beta_n)$ *be two* $n-$*tuples. We say that* α *is* majorized *by* β *and we write* $\alpha \prec \beta$ *if the following system of* $n-1$ *inequalities and one equality is satisfied:*

$$\begin{aligned} \alpha_{(1)} &\leq \beta_{(1)} \\ \alpha_{(1)} + \alpha_{(2)} &\leq \beta_{(1)} + \beta_{(2)} \\ ... &\leq ... \\ \alpha_{(1)} + \alpha_{(2)} + ..\alpha_{(n-1)} &\leq \beta_{(1)} + \beta_{(2)} + \beta_{(n-1)} \\ \alpha_{(1)} + \alpha_{(2)} + .. + \alpha_{(n)} &= \beta_{(1)} + \beta_{(2)} + .. + \beta_{(n)} \end{aligned}$$

We say that α *is* weakly sub-majorized *by* β *and we write* $\alpha \prec_w \beta$ *if all relations above hold with weak inequality.*

Theorem 2 *Consider two distributions of agents' types* F *and* G *such that* $\mu_F = \mu_G = \mu$ *and such that* F *second-order stochastically dominates* G *(in particular* F *has a lower variance than* G*). Then it holds that:*

1. $\forall k, t,\ \sum_{i=1}^{k} y_i^F(t) \leq \sum_{i=1}^{k} y_i^G(t)$

2. For any time t *and for any set of available objects at* t, $\Pi_t \neq \emptyset$*, the expected welfare in the efficient dynamic allocation under* F *is lower than that under* G.

Proof. See Appendix. ■

The benefits of increased variability has also been discussed in the one-object model analyzed in the job search literature. The proof of the above Theorem proceeds by showing that the efficient cutoffs of one distribution are majorized by those of the other[11]. A main application of the above Theorem follows: For a constant arrival rate, the system of differential equations that characterizes the efficient dynamic allocation can be solved explicitly for any number of objects if the distribution of the agents' types is exponential (see Example above), while this is rarely the case for other distributions. Together with the above result, that solution can be used to bound the optimal policy and the associated welfare for large, non-parametric classes of distributions that are often used in applications.

Definition 2 *A non-negative random variable* X *is said to be* new better than used in expectation - *NBUE (*new worse than used in expectation - *NWUE) if*

$$E[X - a \mid \ X > a] \leq (\geq)\ E[X]\ , \forall a \geq 0$$

The classes of NBUE (NWUE) distributions are large and contain most of the distributions that appear in applications. For example, any distribution with an *increasing failure (or hazard) rate* is NBUE, while any distribution with a *decreasing failure rate* is NWUE.

Corollary 1 *Let* F*, the distribution of agents' types be NBUE (NWUE) with mean* μ*. Then, for any* t *and* $\Pi_t \neq \varnothing$*, the expected welfare in the efficient dynamic allocation under* F *is lower (higher) than that under the exponential distribution* $G(x) = 1 - e^{-\frac{x}{\mu}}$.

[11] This Theorem has an analogous counterpart for the cutoffs $z_i(t)$ appearing later in the scenario where allocations can be delayed till the deadline. The proof uses then a recent result about majorization of mean order statistics due to De La Cal and Carcamo[5].

Proof. The result follows directly from Theorem 2 by noting that F second order stochastically dominates $G(x) = 1 - e^{-\frac{x}{\mu}}$ (is second-order stochastically dominated by $G(x) = 1 - e^{-\frac{x}{\mu}}$) is equivalent to F being NBUE (NWUE) . This is Theorem 8.6.1 in Ross [21]. In other words,

$$\forall y \geq 0, \int_y^\infty (1 - F(x))dx \leq (\geq)\mu e^{-\frac{x}{\mu}} \text{ if } F \text{ is NBUE (NWUE)}$$

■

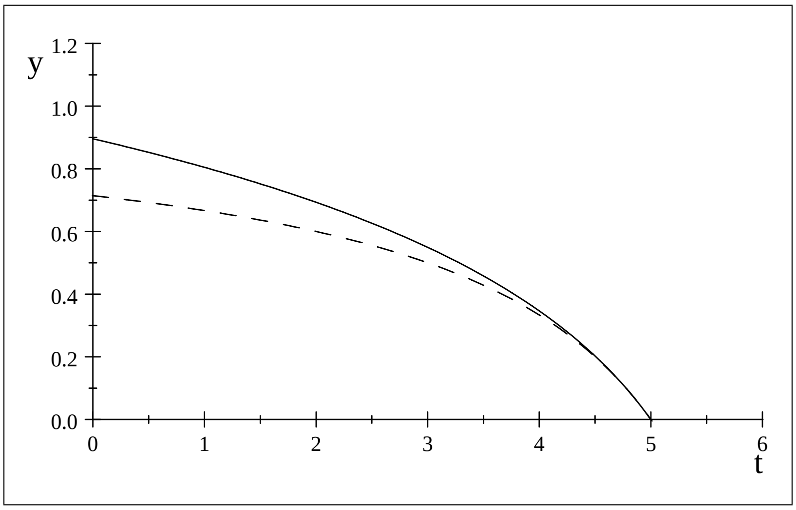
Example 2 *To illustrate the result above, let $F(x) = x$ on $[0,1]$ so that F is IFR and thus NBUE, and let $\lambda(t) = \lambda = 1$. Assume that there is one object with quality $q_1 = 1$. The optimal cut-off curve satisfies*

$$y_F' = -\int_{y_F}^1 (1 - x)dx = -\frac{1}{2} + y_F - \frac{y_F^2}{2}$$

with initial condition $y_F(T) = 0$. The solution to this differential equation is

$$y_F(t) = 1 - \frac{2}{T - t + 2}$$

which is compared in the picture to $\frac{1}{2}\ln[1 + T - t]$, the efficient cutoff curve for the exponential distribution $G(x) = 1 - e^{-2x}$ with mean $\frac{1}{2}$.



Full line: exponential distribution; Dashed line: uniform distribution

4.1.1 The Welfare Loss Versus a Delayed Allocation

Assume that at time t there are n objects left, with qualities $q_1 \geq q_2 ... \geq q_n$. We assume here for simplicity that arrivals follow a homogenous Poisson process with rate λ[12]. Instead of the original formulation, consider now the scenario where the allocation decision to all subsequently arriving agents can be made at time T (the deadline) where their precise number is known, and where their types can be extracted via the usual, static VCG payments. At time T a welfare maximizing designer will allocate the object with the highest quality to the agent with the highest type, the object with the second highest quality to the agent with the second highest type, and so on... (assortative matching). This means that expected welfare at time t is given by $\sum_{i=1}^{n} q_i z_i(t)$, where $z_i(t)$ represents the expected type of an agent who arrives after t, and who get assigned to the object with the $i-th$ highest quality.

In order to calculate this term, let $X_{(i,l)}$ denote the $i-th$ order statistic out of l copies of X, and denote by $\mu_{i,l}$ its expectation (note that $\mu_{l.l}$ is the expectation of the maximum or highest order statistic). Denote by $\Pr_l(t) \geq 0$ the probability that there will be l arrivals, $l \geq 1$, after time t. For a direct comparison to sequential assignment and Poisson arrival process, we need to set:

$$\Pr_l(t) = e^{-\lambda(T-t)} \frac{\lambda^l (T-t)^l}{l!}.$$

Given assortative matching at time T, we then obtain[13]:

$$z_i(t) = \sum_{l=i}^{\infty} \Pr_l(t) \mu_{l-i+1,l} = e^{-\lambda(T-t)} \sum_{l=i}^{\infty} \frac{\lambda^l (T-t)^l \mu_{l-i+1,l}}{l!}, \; i = 1, 2, ..n$$

Note also that

$$\begin{aligned} \sum_{i=1}^{m} z_i(t) &= \sum_{l=1}^{\infty} \Pr_l(t) \left(\sum_{i=1}^{\min(l.m)} \mu_{l-i+1,l} \right) \\ &= e^{-\lambda(T-t)} \sum_{l=1}^{\infty} \frac{\lambda^l (T-t)^l}{l!} \left(\sum_{i=1}^{\min(l.m)} \mu_{l-i+1,l} \right), \; m = 1, 2, ..n \end{aligned}$$

[12] The argument is easily extended to non-homogeneous processes.

[13] Note that an object remains unassigned if there are not sufficient arrivals, yielding a zero reward.

The next result establishes a relation between the $n-$vector of optimal dynamic cutoffs $\{y_i(t)\}_{i=1}^n$ and the n-vector of the corresponding static expected types $\{z_i(t)\}_{i=1}^n$. Intuitively, $\sum_{i=1}^n q_i[y_i(t) - z_i(t)]$ measures the welfare loss due to the sequentiality constraint.

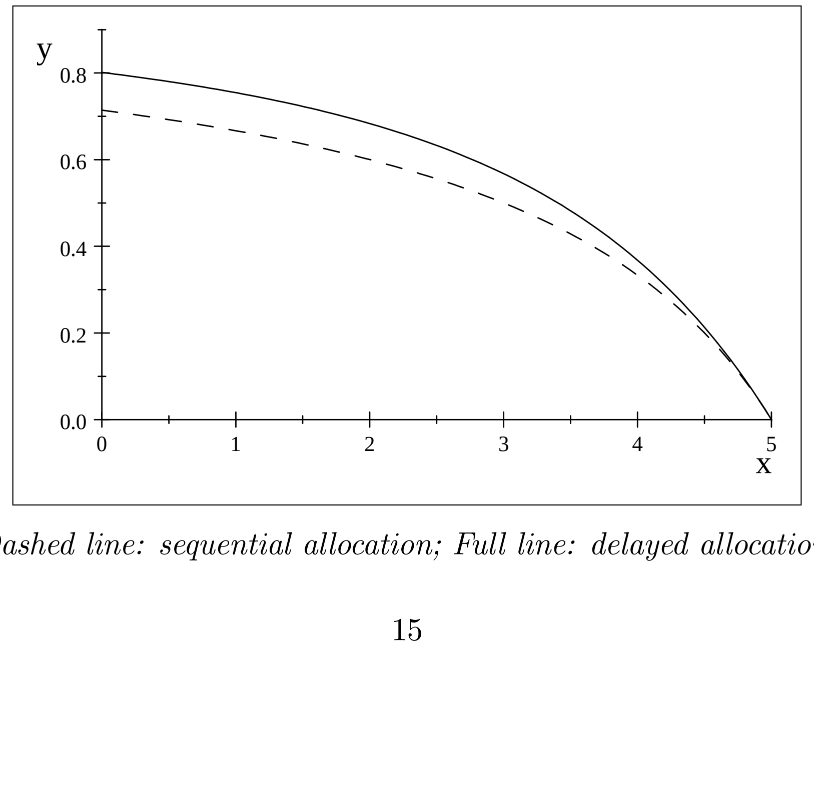
Theorem 3 *For any period t, and for any n, the vector $\{y_i(t)\}_{i=1}^n$ of optimal cutoffs in the welfare maximizing sequential allocation of n objects to agents arriving according to a Poisson process with parameter λ is weakly sub-majorized by the vector $\{z_i(t)\}_{i=1}^n$. Moreover, $\lim_{n\to\infty} \sum_{i=1}^n y_i(t) = \lim_{n\to\infty} \sum_{i=1}^n z_i(t) = \lambda(T-t)\mu$, where μ is the mean of the distribution of agents' types.*

Proof. See Appendix. ■

If the number of homogenous objects n is large, sequentiality does not cause any welfare loss since any arriving agent should get an object. The limit expression is intuitive: for each arrival, expected welfare is given by the average type μ, and $\lambda(T-t)$ is the expected number of arrivals.

Example 3 *For an illustration, let $F(x) = x$ on $[0,1]$, let $\lambda = 1$ and $T = 5$. Assume that there is one object with quality $q_1 = 1$. The optimal cut-off curve for the sequential allocation is given by $y_1(t) = 1 - \frac{2}{7-t}$. Noting that the expectation of the highest order statistic out l uniformly distributed random variables is $\frac{l}{l+1}$, we obtain that*

$$z_1(t) = e^{-(5-t)} \sum_{l=1}^{\infty} \frac{(5-t)^l l}{(l+1)!} = 1 - \frac{1-e^{-(5-t)}}{5-t}$$



Dashed line: sequential allocation; Full line: delayed allocation.

4.2 The Dynamic Efficient Allocation with an Infinite Horizon and Discounting

In this Section we assume that $r(t) = e^{-\alpha t}$. Given this specification, the arrival process can be more general, and it is assumed here to be a *renewal* process with general inter-arrival distribution B (instead of a Poisson process where the inter-arrival distribution is exponential).[14] We start with a simple example that illustrates the main insight: the stationarity of the welfare maximizing dynamic policy.

Example 4 *Let the arrival process be Poisson with rate λ, i.e., $B(t) = 1 - e^{-\lambda t}$, and let $\widetilde{B}$ denote the Laplace- transform of the inter-arrival distribution B. Note first that*[15]

$$\widetilde{B}(\alpha) = \int_0^\infty e^{-\alpha t} \lambda e^{-\lambda t} dt = \frac{\lambda}{\alpha + \lambda}; \quad \frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} = \frac{\lambda}{\alpha}$$

Consider now the the differential equation defining the efficient allocation curve for the case of one object $y_1(t)$(see Theorem 1):

$$\frac{d[r(t)y_1(t)]}{dt} = -\lambda r(t) \int_{y_1}^\infty (1 - F(x))dx$$

Plugging $r(t) = e^{-\alpha t}$ we get

$$(y_1' - \alpha y_1) = -\lambda \int_{y_1}^\infty (1 - F(x))dx.$$

Postulating now $y_1' = 0$ yields

$$y_1 = \frac{\lambda}{\alpha} \int_{y_1}^\infty (1 - F(x))dx = \frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} \int_{y_1}^\infty (1 - F(x))dx$$

The equation above has a unique solution y_1^* since its right hand side decreases in y_1 from $\frac{\lambda}{\alpha}\mu$ (where μ is the mean of F) to 0, and left hand side increases from 0 to infinity. Since $\lim_{t \to \infty} e^{-\alpha t} y_1^* = 0$, we obtain that the efficient dynamic cut-off curve is indeed described by the constant y_1^*. The derivations for more items follow analogously.

[14] The derived controlled stochastic process is *semi-Markov* since the Markov property is preserved only at decision points, but not between them. See Puterman (2005) for solution approaches to such problems by an *uniformization* procedure, and for conditions guaranteeing that optimal policies are deterministic and Markovian.

[15] $\widetilde{B}(\alpha)$ acts here as the effective discount rate. It represents the discounted value of one unit at the expected time of the next arrival.

The complete-information efficient dynamic assignment for the general case is characterized in the following Theorem:

Theorem 4 *(Albright, [1]) Assume that $r(t) = e^{-\alpha t}$. The efficient allocation curves are constants (i.e., independent of time) $y_n \leq y_{n-1} ... \leq y_1$. These constants do not depend on the q's, and are given by the implicit recursion:*

$$(y_k + y_{k-1} + ..y_1) = \frac{\widetilde{B}(\alpha)}{1 - \widetilde{B}(\alpha)} \int_{y_k}^{\infty} (1 - F(x))dx \ , \ 1 \leq k \leq n$$

where $\widetilde{B}$ is the Laplace- transform of the inter-arrival distribution B.

The efficient dynamic allocation policy is obviously Markovian and deterministic, and can be therefore implemented by the payments of Proposition 1. The analog of Theorem 2 for this case is:

Theorem 5 *Consider two distributions of agents' types F and G such that $\mu_F = \mu_G = \mu$ and such that F second-order stochastically dominates F (in particular F has a lower variance than G). Then, for any fixed inter-arrival distribution B it holds that:*

1. *$\forall k$, $\sum_{i=1}^{k} y_i^F \leq \sum_{i=1}^{k} y_i^G$*
2. *For any t and any $\Pi_t \neq \emptyset$ the expected welfare in the efficient dynamic allocation under F is lower than that under G.*

Proof. See Appendix. ■

In addition to the above Theorem about the benefits of increased variability in the agents' types, we now obtain a comparative-statics result about the benefits of variability in arrival times. Interestingly, this next result holds for a stochastic order that is much weaker than second-order stochastic dominance. We first need the following definition (see Shaked and Shanthikumar, 2007):

Definition 3 *Let X, Y be two non-negative random variables. Then X is said to be smaller than Y in the Laplace transform order, denoted by $X \leq_{Lt} Y$, if*

$$E[e^{-sX}] \geq E[e^{-sY}] \ \text{ for all } s > 0$$

The function $w(x) = -e^{-sx}$ is increasing an concave for any $s > 0$. Thus, we obtain that $X \leq_{SSD} Y \Rightarrow X \leq_{Lt} Y$ since the former involves a comparison of expectations with respect to **all** increasing concave functions.

Theorem 6 *Consider two inter-arrival distributions B and E such that $B \geq_{Lt} E$. Then, for any fixed distribution of agents' characteristics F, it holds that:*

1. $\forall k,\ \sum_{i=1}^{k} y_i^B \leq \sum_{i=1}^{k} y_i^E$

2. *For any t and for any $\Pi_t \neq \emptyset$, the expected welfare in the efficient dynamic allocation under B is lower than that under E.*

Proof. See Appendix. ■

The above result is in sharp contrast with a result from the job search literature: in case of a constant per-period cost of search, a higher variance of the number of offers per period is detrimental.[16] Again, we can apply the above comparative static results in order to bound the optimal cut-off curves and the associated expected welfare for large classes of distributions of the agents' types and of the inter-arrival times.

Corollary 2 *For any t and for any $\Pi_t \neq \emptyset$ we have:*

1. *For any fixed distribution of inter-arrival times, the expected welfare under an NBUE(NWUE) distribution of agents' types with mean μ is lower (higher) than the expected welfare under the exponential distribution $G(t) = 1 - e^{-\frac{t}{\mu}}$.*

2. *For any fixed distribution of agents' types, the expected welfare under an NBUE(NWUE) distribution of inter-arrival times with mean μ is lower (higher) than the expected welfare under a Poisson arrival process with rate $\frac{1}{\mu}$.*

Proof. The first claims follows from Theorems 5 and from the fact that NBUE (NWUE) distributions second order stochastically dominate (are dominated by) an exponential distribution with the same mean (see also the proof of Corollary 1). The second claim follows from Theorem 6, from the above observation, and from the fact that second order stochastic dominance implies domination in the Laplace-transform order. ■

[16] See Lippman and McCall [13], among others.

Example 5 *Assume that there is one object with* $q_1 = 1$, *let the discount rate be* α, *and consider the exponential distributions* $G_\mu = 1 - e^{-\frac{t}{\mu}}$ *for the agents' types and* $G_\omega = 1 - e^{-\frac{t}{\omega}}$ *for the inter-arrival times. For these distributions, the optimal cutoff point* y_1 *solves*

$$y_1 = \frac{\widetilde{G}_\omega(\alpha)}{1 - \widetilde{G}_\omega(\alpha)} \int_{y_1}^{\infty} e^{-\frac{t}{\mu}} dt = \frac{\mu}{\omega\alpha} e^{-\frac{y_1}{\mu}}$$

The solution to this equation is given by

$$y_1 = \mu \, \mathrm{LambertW}(\frac{1}{\omega\alpha})$$

where the increasing function $\mathrm{LambertW}(x)$ *is implicitly defined by*

$$\mathrm{LambertW}(x) e^{\mathrm{LambertW}(x)} = x$$

Thus, the expected welfare under the efficient policy is lower (higher) than $\mu \, \mathrm{LambertW}(\frac{1}{\omega\alpha})$ *if the distribution of agents' abilities is NBUE(NWUE) with mean* μ, *and if the distribution of inter-arrival times is NBUE(NWUE) with mean* ω.

5 Budget Balanceness

The above analysis was concerned solely with physical allocational efficiency. Moreover, the employed mechanisms were individual rational. While keeping these two requirements, we need to deal with another aspect: the monetary budget deficit/surplus that the mechanism generates. We assume below that there are no alternative resources to subsidize the allocation process, and hence we require that the total payment is non-negative. But, an inability to redistribute the raised money among agents reduces their welfare and hence prevents reaching a fully efficient outcome. Roughly speaking, in an allocatively efficient mechanism, the payment of a buyer is tied to the externality he imposes on other agents, and hence it depends on the values of others. A redistribution of this payment to others is therefore bound to affect their incentives. In our dynamic setting, budget balancedness is easier to achieve due to the sequential nature of the allocation process: the amount each agent pays depends on the imposed *expected* externality, and hence it is not affected by the realized reports of the agents that should get this amount as a refund.

We propose below mechanisms that reallocate the gathered money among the agents without distorting their incentives.[17] We consider two classes of mechanisms: 1) *Online mechanisms*, where monetary transfers pertaining to a specific transaction need to be completed together with the physical allocation; 2) *Offline mechanisms*, where payments can be postponed to later stages. For simplicity, we restrict attention to a homogeneous, Poisson arrival process with parameter λ.

5.1 Online mechanisms

Denote by $B(t)$ the budget surplus accumulated by time t when using a mechanism that implements the efficient physical allocation and never runs a deficit (thus $B(t) \geq 0,\ \forall t$). While it is impossible here to precisely obtain ex-post budget balancedness, we describe below a scheme where the expected surplus vanishes asymptotically, as $\lambda \longrightarrow \infty$.

Definition 4 (Online redistribution mechanism) *A scheme that implements the efficient allocation such that the type-independent part of the payment scheme for the agent arriving at t is $S(t) = -B(t)$ will be called a* dynamically efficient online redistribution *mechanism (DEON hereafter).*

In the *DEON* mechanism, if an agent arrives at time t and reports type $x_t \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$, while the previous agent arrived at $\tau < t$ and reported a type $x_\tau \in [y_{l,\Pi_\tau}(\tau), y_{l-1,\Pi_\tau}(\tau))$ then the time t agent's payment is given by

$$\sum_{i=j}^{k_t}(q_{(i)} - q_{(i+1)})y_{i,\Pi_t}(t) - \sum_{i=l}^{k_\tau}(q_{(i)} - q_{(i+1)})y_{i,\Pi_\tau}(\tau).$$

That is, each agent's payment consists of the two parts: first, a non-negative fee for the object he gets, which is determined by the report and by the efficient cutoffs; second, a refund equal to the payment of the previously arrived agent.

Clearly, the *DEON* mechanism never runs a budget deficit, but it may run a surplus if, after the last sale, no agent arrives till the deadline. Since the probability that no

[17] We restrict attention here to the more interesting deadline scenario. The results can be easily extended to the infinite horizon case.

agent arrives between periods s and T is $e^{-\lambda(T-s)}$, the expected surplus of the mechanism is given by

$$ES(T) = \sum_{\Pi_s \subseteq \Pi} \int_0^T \Pr_s(\Pi_s) \sum_{j=1}^{k_s} \sum_{i=j}^{k_s} (q_{(i,\Pi_s)} - q_{(i+1,\Pi_s)}) y_{i,\Pi_s}(s)\, g_{i,\Pi_s}(s)\, e^{-\lambda(T-s)} ds$$

where $\Pr_\tau(\Pi_s)$ is the probability that at time τ the set of the objects still available is Π_s, $q_{(j,\Pi_s)}$ is the j'th highest quality out of Π_s, and $g_{i,\Pi_s}(s)$ is the density that at time s an agent arrives with a type that leads to the allocation of the j'th highest quality out of Π_s.

The next result shows that the expected surplus $ES(T)$ goes to zero if the rate of the Poisson process goes to infinity, that is if $\lambda \to \infty$. Note that increasing λ has two opposite effects on the expected surplus: On the one hand, prices and the expected surplus in case no agent arrives after the last sale go up. On the other, the probability of no arrival after the last sale goes down. The next Theorem shows that the second effect dominates. Since λ governs the number of arrivals per uit of time, a similar result holds if λ is constant but $T \to \infty$.

Theorem 7 *For any distribution of values, and for any deadline* T, $\lim_{\lambda\to\infty} ES(T) = 0$.

Proof. See Appendix. ■

5.2 Offline mechanism

In this subsection we show how to construct a mechanism that satisfies ex-post budget balancedness for any λ and T by relaxing the "online" requirement that all monetary transfers need to be implemented upon arrival. We sketch here the case where there is one object, with quality $q = 1$, but the generalization to the case with several heterogenous objects is straightforward.

While online mechanisms could allocate the accrued surplus from a given sale only among the agents that arrive after that transaction has been conducted (which may lead to a budget surplus if no further arrivals take place), in the current setting the designer can observe the number of the agents that eventually arrive, and may ex-post refund the entire fee to the buyer who paid it if nobody else shows up. But, in order not to distort that buyer's incentives, he has to pay a higher net price if additional agents subsequently arrive.

Consider a time t where the object is still available, and the direct mechanism where: 1) the arriving agent at t does not gets the object and pays nothing if his reported type is below $y_1(t)$; 2) the arriving agent at t gets the object and pays $P(t)$ if his reported type is above $y_1(t)$. Moreover, at time T, the designer distributes the raised revenue equally among all agents that arrived after the sale. In other words, if a sale occurs at t and no agents arrive after t, the buyer gets his entire payment back; if one agent arrived after t, the buyer gets half of his payment back, and so on...Obviously, this mechanism is budget-balanced.

The expected utility of an agent that arrives at t (if the object is still available) with type x who reports a type above $y_1(t)$ is given by

$$x - P(t) + \sum_{l=0}^{\infty} \Pr_t(l) \frac{P(t)}{1+l}$$

where $\Pr_t(l)$ is the probability that exactly l additional agents arrive during the time period between t and T. Since the arriving process is Poisson, we have

$$\sum_{l=0}^{\infty} \frac{\Pr_t(l)}{1+l} = \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)}.$$

In order to implement efficient allocation, the type $y_1(t)$ should be exactly indifferent between getting the object, which yields utility $x - P(t) + \sum_{l=0}^{\infty} \Pr_t(l) \frac{P(t)}{1+l}$, and not getting the object, which generates utility of zero (recall that only buyers arriving after a sale may get some redistributed revenue). This indifference implies that the payment $P(t)$ must be given by

$$P(t) = \frac{y_1(t)}{1 - \frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)}} = \frac{\lambda(T-t)\, y_1(t)}{\lambda(T-t) + e^{-\lambda(T-t)} - 1}.$$

Although the proposed mechanism is interim individually rational, it is not be individually rational ex-post. It is impossible to attain ex-post individual rationality if the designer insists on an efficient allocation and on budget balancedness. To see this, observe that if a buyer is the only arriving agent, he should ultimately pay zero (by budget-balancedness); since, by efficiency, the type $y_1(t)$ should be indifferent between buying and not, we obtain that sometimes this type needs to pay strictly more than his valuation.

6 Conclusion

We have studied the continuous-time, sequential, welfare-maximizing assignment of several heterogenous objects to privately informed agents that arrive according to a Poisson or renewal process. Our results yield upper/lower bounds on efficiency for large classes of distributions of agents' characteristics or of distributions of inter-arrival times for which explicit solutions cannot be obtained in closed form. We have also compared welfare in our setting to other scenarios where physical or monetary allocations can be delayed until more information becomes available.

The analysis was made possible by a fruitful combination of insights from stochastic dynamic programming, mathematical statistics, and mechanism design. We believe that the applicability of a large class of interesting dynamic problems that have been extensively studies in the Economics, Operations Research and Management Science literature can be significantly increased by adding to their structure information and incentive considerations.

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7 Appendix

First, we prove a Lemma that is used in the proof of Theorem 2.

Lemma 1 *Let* $\alpha = (\alpha_1, \alpha_2, ..\alpha_n)$ *and* $\beta = (\beta_1, \beta_2, .., \beta_n)$ *be two* $n-$*tuples such that* $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$. *Then* $\alpha \prec \beta$ *if and only if* $\sum_{i=1}^{n} q_i \alpha_{(i)} \leq \sum_{i=1}^{n} q_i \beta_{(i)}$ *for any constants* $q_n \leq q_{n-1} \leq ... \leq q_1$.

Proof of Lemma 1. $\Longleftarrow$ Assume that $\sum_{i=1}^{n} q_i \alpha_{(i)} \leq \sum_{i=1}^{n} q_i \beta_{(i)}$ for any $q_n \leq q_{n-1} \leq ... \leq q_1$. For each $k = 1, 2, .., n-1$ consider $q^k = (q_1^k, q_2^k, .., q_n^k)$ where $q_i^k = 1$ for $i = 1, 2, ..k$, and $q_i^k = 0$ for $i = k+1, k+2, ..n$. Then, for each k we obtain

$$\sum_{i=1}^{n} q_i^k \alpha_{(i)} \leq \sum_{i=1}^{n} q_i^k \beta_{(i)} \Leftrightarrow \sum_{i=1}^{k} \alpha_{(i)} \leq \sum_{i=1}^{k} \beta_{(i)}$$

and thus $\alpha \prec \beta$.

$\Longrightarrow$ Assume $\alpha \prec \beta$ and let $q_n \leq q_{n-1} \leq ... \leq q_1$. Then we have the following chain:

$$\begin{aligned}
\sum_{i=1}^{n} q_i[\beta_{(i)} - \alpha_{(i)}] &= q_n \sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] + \sum_{i=1}^{n-1}(q_i - q_n)[\beta_{(i)} - \alpha_{(i)}] \\
&= q_n \sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] + (q_{n-1} - q_n)\sum_{i=1}^{n-1}[\beta_{(i)} - \alpha_{(i)}] \\
&\quad + \sum_{i=1}^{n-2}(q_i - q_{n-1})[\beta_{(i)} - \alpha_{(i)}] \\
&= ... \\
&= q_n \sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] + \sum_{j=1}^{n-1}(q_j - q_{j+1})(\sum_{i=1}^{j}[\beta_{(i)} - \alpha_{(i)}]) \geq 0
\end{aligned}$$

The last inequality follows since: 1. $\sum_{i=1}^{n}[\beta_{(i)} - \alpha_{(i)}] = \sum_{i=1}^{n}\beta_i - \sum_{i=1}^{n}\alpha_i = 0$ by definition; 2. $\forall j$, $q_j - q_{j-1} \geq 0$ by definition; 3. $\forall j$, $\sum_{i=1}^{j}[\beta_{(i)} - \alpha_{(i)}] \geq 0$ by majorization.

■

Proof of Theorem 2. 1. By Theorem 1 we know that

$$\begin{aligned}
\frac{d(\sum_{i=1}^{k} y_i^F(t))}{dt} &= -\lambda(t)\int_{y_k^F(t)}^{\infty}(1 - F(x))dx \\
\frac{d(\sum_{i=1}^{k} y_i^G(t))}{dt} &= -\lambda(t)\int_{y_k^G(t)}^{\infty}(1 - G(x))dx
\end{aligned}$$

Define first: $H_F(s) = \int_s^{\infty}(1 - F(x))dx$ and $H_G(s) = \int_s^{\infty}(1 - G(x))dx$. These are both positive, decreasing functions with $H_F(0) = H_G(0) = \mu$.

By SSD, for any $s \geq 0$ it holds that

$$\begin{aligned}
\int_0^s F(x))dx &\leq \int_0^s G(x))dx \Leftrightarrow \int_0^s (1 - F(x)))\,dx \geq \int_0^s (1 - G(x)))\,dx \\
&\Leftrightarrow \int_s^{\infty}(1 - F(x))dx \leq \int_s^{\infty}(1 - G(x))dx \Leftrightarrow H_F(s) \leq H_G(s)
\end{aligned}$$

where the second line follows because

$$\int_0^{\infty}(1 - F(x))dx = \mu_F = \mu_G = \int_0^{\infty}(1 - G(x))dx.$$

Thus, the curve H_F is always below H_G.

Consider now $y_1^F(t)$ and $y_1^G(t)$. These are, respectively, the solutions to the differential equations :

$$y' = -\lambda(t)H_F(y) \;\text{ and } y' = -\lambda(t)H_G(y)$$

with boundary condition $y(T) = 0$. Integrating the above equations from t to T, and using the boundary condition, we get the integral equations:

$$\begin{aligned} y(T) - y(t) &= -\int_t^T \lambda(s)H_F(y(s))ds \Leftrightarrow y(t) = \int_t^T \lambda(s)H_F(y(s))ds \text{ and} \\ y(T) - y(t) &= -\int_t^T \lambda(s)H_G(y(s))ds \Leftrightarrow y(t) = \int_t^T \lambda(s)H_G(y(s))ds \end{aligned}$$

Because H_F is always below H_G and because these are decreasing functions, we obtain $y_1^F(t) \leq y_1^G(t)$.

Consider now $y_1^F(t)+y_2^F(t)$ and $y_1^G(t)+y_2^G(t)$. These functions satisfy the differential equations:

$$y' = -\lambda(t)\int_{y-y_1^F(t)}^{\infty}(1-F(x))dx \text{ and } y' = -\lambda(t)\int_{y-y_1^G(t)}^{\infty}(1-F(x))dx$$

with boundary condition $y(T) = 0$. Integrating from t to T yields the equations:

$$\begin{aligned} y(t) &= \int_t^T \lambda(s)\left[\int_{y(s)-y_1^F(s)}^{\infty}(1-F(x))dx\right] ds = \int_t^T \lambda(s)H_F[(y(s)-y_1^F(s)] \, ds \\ y(t) &= \int_t^T \lambda(s)\left[\int_{y(s)-y_1^G(s)}^{\infty}(1-G(x))dx\right] ds = \int_t^T \lambda(s)H_G[(y(s)-y_1^G(s)] \, ds \end{aligned}$$

We have :

$$\forall t, \; H_F[(y(t)-y_1^F(t)] \leq H_F[(y(t)-y_1^G(t)] \leq H_G[(y(t)-y_1^G(t)]$$

where the first inequality follows because $y_1^F(t) \leq y_1^G(t) \Leftrightarrow y(t) - y_1^F(t) \geq y(t) - y_1^G(t)$ and because the function H_F is decreasing, and the second inequality follows because H_F is always below H_G. This yields $y_1^F(t) + y_2^F(t) \leq y_1^G(t) + y_2^G(t)$, as required. The rest of the proof follows analogously.

2. The expected welfare terms from time t on if there are k objects left are given by $\sum_{i=1}^k q_{(i)}y_i^F(t)$ and by $\sum_{i=1}^k q_{(i)}y_i^G(t)$, respectively. By point 1, we know that for each k and for each t, $y^{kF}(t) = (y_1^F(t), y_2^F(t), ..y_k^F(t)) \prec_w (y_1^G(t), y_2^G(t), ..y_k^G(t)) := y^{kG}(t)$. By Result 12.5 (b) in Pecaric et. al [18], for each k and each t there exists a $k-$vector $z(t)$ such that $z(t) \prec y^{kG}(t)$ and such that $z_i(t) \geq y_i^{kF}$, $\forall i$. We obtain then:

$$\forall k, t, \;\; \sum_{i=1}^k q_{(i)}y_i^F(t) \leq \sum_{i=1}^k q_{(i)}z_i(t) \leq \sum_{i=1}^k q_{(i)}y_i^G(t)$$

where the last inequality follows from Lemma 1. ■

Proof of Theorem 3. Consider a situation with $m \leq n$ identical objects available at time t. For any particular realization of arrivals and agents types from period t on, welfare in the case where the decision can be delayed until the entire realization has been observed is given by $\sum_{i=1}^{m} z_i(t)$. Consider now the original sequential assignment problem where agents arrive according to a Possion process, and allocations must be made upon arrival. By Theorem 1, the expected total welfare is then given by $\sum_{i=1}^{m} y_i(t)$. It is obvious that, for any realization of arrivals and types, this total reward cannot be larger than that obtained in the previous scenario where allocations can be delayed since mistakes may have occurred due to the constraints of sequentiality, i.e., an agent with a lower type was served although there was another arrival with a higher type[18]. This yields $\sum_{i=1}^{m} y_i(t) \leq \sum_{i=1}^{m} z_i(t)$, $\forall m \leq n$, as required.

The second part follows by considering the limit when the number of identical objects tends to infinity. In that limit, all arriving agents should be served, and there is no welfare loss due to the sequentiality constraint. By Theorem 1 we know that

$$\frac{d[\sum_{i=1}^{n} y_i(s)]}{ds} = -\lambda \int_{y_n(s)}^{\infty} (1 - F(x))dx$$

Integrating the above expression between t and T yields:

$$\begin{aligned}
\sum_{i=1}^{n} y_i(T) - \sum_{i=1}^{n} y_i(t) &= -\lambda \int_t^T \left(\int_{y_n(s)}^{\infty} (1 - F(x))dx \right) ds \Leftrightarrow \\
\sum_{i=1}^{n} y_i(t) &= \lambda \int_t^T \left(\int_{y_n(s)}^{\infty} (1 - F(x))dx \right) ds
\end{aligned}$$

Taking the limit with respect to n, we finally obtain:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} y_i(t) &= \lim_{n\to\infty} \lambda \int_t^T \left(\int_{y_n(s)}^{\infty} (1 - F(x))dx \right) ds \\
&= \lambda \int_t^T \left(\int_0^{\infty} (1 - F(x))dx \right) ds = \lambda(T - t)\mu
\end{aligned}$$

where the second line follows since $\lim_{n\to\infty} y_n(t) = 0$ uniformly. ■

[18] In other words, any allocation obtained in the sequential process can be replicated when delayed allocations are possible.

Proof of Theorem 5. **1.** Define first $H_F(s) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_s^\infty (1-F(x))dx$ and $H_G(s) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_s^\infty (1-G(x))dx$. These are both decreasing functions and

$$H_F(0) = H_G(0) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)}\mu$$

Consider now y_1^F and y_1^G. These are, respectively, the solutions to the equations:

$$s = H_F(s) \text{ and } s = H_G(s)$$

By SSD, for any $s \geq 0$ it holds that

$$\begin{aligned}\int_0^s F(x))dx &\leq \int_0^s G(x))dx \Leftrightarrow \int_0^s (1-F(x)))\,dx \geq \int_0^s (1-G(x)))\,dx \\ &\Leftrightarrow \int_s^\infty (1-F(x))dx \leq \int_s^\infty (1-G(x))dx \Leftrightarrow H_F(s) \leq H_G(s)\end{aligned}$$

Thus, the decreasing curve $H_F(s)$ is always below the decreasing curve $H_G(s)$ and we obtain $y_F^1 \leq y_G^1$. Consider now y_2^F and y_2^G which are defined by the equations:

$$\begin{aligned} y_2^F + y_1^F &= \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_{y_2^F}^\infty (1-F(x))dx \\ y_2^G + \ y_1^G &= \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \int_{y_2^G}^\infty (1-G(x))dx \end{aligned}$$

Equivalently, $y_2^F + y_2^F$ and y_2^G $+y_1^G$ are, respectively, the solutions of:

$$s = H_F(s - y_1^F) \text{ and } s = H_G(s - y_1^G)$$

Recalling that $y_1^F \leq y_1^G$, we obtain $s - y_1^F \geq s - y_1^G$,$\forall s$. This yields:

$$H_F(s - y_1^F) \leq H_F(s - y_1^G) \leq H_G(s - y_1^G)$$

where the first inequality follows because the function H_F is decreasing, and the second inequality follows by SSD. Thus, the curve $H_F(s - y_1^G)$ is always below the curve $H_G(s - y_1^G)$ and the result follows as above. The rest of the proof is completely analogous.

2. The expected welfare terms from time t on if k objects left are given by $e^{-\alpha t}\left[\sum_{i=1}^k q_{(i)} y_i^F\right]$ and by $e^{-\alpha t}\left[\sum_{i=1}^k q_{(i)} y_i^G\right]$, respectively. The proof proceeds exactly as that of Theorem 2-2. ■

Proof of Theorem 6. **1.** Let $H_B(s) = \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)}\int_s^{\infty}(1-F(x))dx$, $H_E(s) = \frac{\widetilde{E}(\alpha)}{1-\widetilde{E}(\alpha)}\int_s^{\infty}(1-F(x))dx$ where $\widetilde{B}$ and $\widetilde{E}$ are the respective Laplace transforms. By the definition of the Laplace transform, and by the assumption $B \geq_{Lt} E$, we know that $\widetilde{B}(\alpha) \leq \widetilde{E}(\alpha)$. This yields:

$$\widetilde{B}(\alpha) \leq \widetilde{E}(\alpha) \Leftrightarrow \frac{\widetilde{B}(\alpha)}{1-\widetilde{B}(\alpha)} \leq \frac{\widetilde{E}(\alpha)}{1-\widetilde{E}(\alpha)} \Leftrightarrow H_B(s) \leq H_E(s)$$

The first equivalence follows because the function $\frac{x}{1-x}$ is increasing on the interval $[0,1)$ with $\lim_{x\to 1}\frac{x}{1-x} = \infty$, and because Laplace transforms take values in the interval $[0,1]$.

Thus, we obtained that the decreasing function $H_B(s)$ is always below the decreasing function $H_E(s)$. Consider first y_1^B and y_1^E. These are, respectively, the solutions to the equations

$$s = H_B(s) \text{ and } s = H_E(s)$$

The rest of the proof continues analogously to the proof of Theorem 5-1.

2. This follows analogously to the proof of Theorem 2-2. ∎

Proof of Theorem 7. Let $k_0 = 1$. That is, initially there is only one object available of quality q normalized to be one. The expected surplus of the *DEON* mechanism is given by

$$\int_0^T y_1(t)\,\lambda\left[1-F\left(y_1(t)\right)\right] e^{-\lambda\int_0^t [1-F(y_1(z))]dz} e^{-\lambda(T-t)}dt$$

$$= \lambda\frac{y_1(t)\,\lambda\left[1-F\left(y_1(t)\right)\right] e^{\lambda\int_0^t F(y_1(z))dz}}{e^{\lambda T}}ds$$

Since $\left[1-F\left(y_1(t)\right)\right] \leq 1$ it is sufficient to show that

$$\lim_{\lambda\to\infty}\int_0^T \frac{\lambda y_1(t)}{e^{\lambda\left(T-t+\int_0^t [1-F(y_1(z))]dz\right)}}dt = 0.$$

The claim will follow by showing that, for any $t \in [0,T]$, the integrand goes to 0 when $\lambda \to \infty$. If $t = T$, the result follows immediately since $y(T) = 0$. Assume then that $t < T$, and recall that

$$y_1'(t) = -\lambda\int_{y_1(t)}^{\infty}\left[1-F(x)\right]dx$$

Integrating both sides yields between t and T yields

$$y_1(t) = \lambda \int_t^T \int_{y_1(s)}^{\infty} [1 - F(x)]\, dxds \leq \lambda (T - t) E(x)$$

where the last inequality follows from $\int_0^{\infty} [1 - F(x)]\, dx = E(x)$. In addition, for any $t < T$, there exists a constant $A > 0$ such that $T - t + \int_0^t [1 - F(y_1(z))]\, dz > A$. Therefore, it is sufficient to show that

$$\lim_{\lambda \to \infty} \frac{\lambda^2 (T - t) E(x)}{e^{\lambda A}} = 0.$$

Applying L'Hospital's rule twice on the above ratio provides the required results.

If $k_0 = 2$ the expected surplus of the *DEON* mechanism is

$$\int_0^T \frac{\lambda((q_1 - q_2) y_1(t) + q_2 y_2(t)) [1 - F(y_1(t))]}{e^{\lambda\left(T - t + \int_0^t [1 - F(y_2(z))]dz\right)}} dt + q_2 \int_0^T \frac{y_2(t) \lambda [F(y_1(t)) - F(y_2(t))]}{e^{\lambda\left(T - t + \int_0^t [1 - F(y_2(z))]dz\right)}} dt +$$

$$q_1 \int_0^T \int_s^T \frac{\lambda^2 y_1(t) [1 - F(y_1(t))] [F(y_1(s)) - F(y_2(s))]}{e^{\lambda\left(T - t + \int_s^t [1 - F(y_1(z))]dz + \int_0^s [1 - F(y_2(z))]dz\right)}} dtds +$$

$$q_2 \int_0^T \int_s^T \frac{\lambda^2 y_1(t) [1 - F(y_1(s))] [1 - F(y_1(t))]}{e^{\lambda\left(T - t + \int_0^s [1 - F(y_2(z))]dz + \int_s^t [1 - F(y_2(z))]dz\right)}} dtds$$

Similarly to the above argument, each element in the summation goes to 0 as λ goes to infinity, and the result follows analogously. ■