Multidimensional Mechanism Design for Auctions with Externalities*

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In an auction with externalities, a buyer's type is multidimensional and specifies the payoff he would get for each of the $N+1$ possible outcomes: the seller keeps the object or buyer i ($i = 1, ..., N$) gets the object. We provide a characterization of multidimensional incentive compatible mechanisms similar to that for one-dimensional mechanisms. Although reservation utilities are endogenous and type-dependent, the participation constraint is binding for only one "critical" type. A main difficulty in a multidimensional setting is the "integrability" condition. We present a geometric characterization for discontinuous conservative vector fields. In auctions where the buyers submit scalar bids and the seller transfers the object to one of the buyers for sure, a second-price auction maximizes revenue. With two buyers, this auction remains optimal even if the seller can set a reservation price. *Journal of Economic Literature* Classification Numbers: C7, C72.   © 1999 Academic Press

1. INTRODUCTION

Mechanism design problems appear naturally in numerous economic situations, including the regulation of a monopolist, auctions, government procurement, nonlinear pricing, and the provision of public goods. By and large, most of the vast literature on mechanism design has been restricted to the case in which uncertainty is modelled by a single parameter, and the commodity space has dimension two (for example, the quantity-money commodity space in nonlinear pricing). In many problems, however, a satisfying analysis requires a multidimensional treatment.

A fundamental assumption in the literature on optimal auction design is that agents' final payoffs are determined solely by whether or not they obtain the auctioned good, and by the payments made as required by the rules of the auction (see, for example, Myerson [13], and Milgrom and Weber [11]). The possibility that the auctioned good might play a role in future interactions among the auction's participants is excluded. But, there are many situations in which an auction's participants interact after the close of the auction, and where the outcome of the auction affects the nature of their future interaction. Then, when a buyer does not obtain the object, he is no longer indifferent about the identity of the winner of the auction. Several illustrations are: changes of ownership (such as mergers or privatizations) in oligopolistic markets; the sale of a patent when there is downstream competition between buyers; the award of major projects that lead to the creation of a new technology standard; the location of environmentally hazardous enterprises such as waste management plants or nuclear reactors; the location of a potentially powerful international organization such as the European Central Bank.

The previous discussion suggests a model that differs from most of the literature on optimal auction design, and more broadly, mechanism design problems in two important respects:

(1) Buyer i's preferences are determined by an N-dimensional vector $t^i = (t_1^i, ..., t_N^i)$. The coordinate t_i^i can be thought of as the usual "private value" of player i, while each other coordinate t_j^i can be interpreted as his total payoff (excluding payments made in the auction) should buyer j get the object. We use the descriptive term "externalities" to refer to these interaction terms. We are then confronted with a multidimensional mechanism design problem.

(2) Since buyers will generally be unable to escape the effect of externalities simply by refusing to participate in the auction, the agents' "reservation utilities" are neither exogenously given, nor type-independent. Consider the event where a buyer refuses to participate in the auction. In contrast to auctions without externalities, where buyers' outside options

can be normalized to zero, when externalities are present, a buyer's reservation value will, in effect, vary with both his type and the seller's action.

In this framework, we first characterize incentive compatible and individually rational auction mechanisms. We then turn to the problem of finding optimal mechanisms that maximize the seller's expected utility (revenue plus externality), in the restricted class of standard auctions (defined below).

There are several papers concerning multidimensional mechanism design. Two strands of this literature are fairly well developed, the first studying nonlinear pricing in multiproduct monopoly, and the second addressing the regulation of multiproduct monopolists. Among the contributions to the former strand are: Champsaur and Rochet [2], Laffont, Maskin and Rochet [5], McAfee, McMillan and Whinston [10], Mirman and Sibley [12], Palfrey [14], Spence [19], and Wilson [20, 21]. Notable contributions to the second strand include: Laffont and Tirole [6], and Lewis and Sappington [7]. In addition, there are a few papers we would like to mention because they concentrate on the development of general tools and methods, and because they illustrate the additional difficulties that are present in multidimensional models. They are: Matthews and Moore [8], McAfee and McMillan [9], Rochet [15], Rochet and Choné [16], and Armstrong [1]. We discuss the last three because they are relevant for our own work.

Rochet [15] introduces a conjugate duality approach, and presents a very useful characterization result for incentive compatible mechanisms. The analysis is restricted to utility functions that are linear in their characteristic space. For a multiproduct monopolist model, Rochet and Choné [16] show that in an optimal mechanism, the participation constraint is active for a set of types (possibly of positive measure).

Armstrong [1] studies the optimal tariff for a multi-product monopolist selling to a measure of non-interacting consumers. He is able to reduce substantially the complexity of the underlying optimization problem, and obtains solutions in closed form for classes of problems in which the agents' utilities and the distribution of their characteristics are related in a specific way. In addition, given some mild technical conditions, Armstrong shows that it is always in the interest of the monopolist to exclude a positive measure of buyers from consumption.

In Jehiel, Moldovanu and Stacchetti [3] we look at a complementary framework where buyers have private information on the externalities they *cause to others*. Moreover, we assume there that these externalities do not depend on the sufferer's identity. The seller's problem in this framework is much simpler, and eventually reduces to a one-dimensional mechanism design problem for which a full solution is obtained.

The paper is organized as follows: In Section 2 we present the model and invoke the revelation principle to formulate the design problem in terms of two functions on the space of type profiles: the probability assignment function, specifying the random allocation of the object amongst the players, and the payment function stipulating the transfer from each buyer to the seller.

In Section 3 we use conjugate duality to derive necessary and sufficient conditions for a mechanism to be incentive compatible, similar to the envelope condition for one-dimensional mechanism design problems. This characterization result allows us to eliminate half the variables (the payment functions) in the problem. In particular, we show that incentive compatible mechanisms produce interim utility functions for each agent which are convex potentials. The gradient of an agent's interim utility is the conditional probability vector given his type, where each coordinate is the probability that the object be assigned to the corresponding buyer. Since i's interim utility is a potential, the conditional probability assignment vector must be a *conservative* function of his type. We informally call these requirements the "integrability constraints".

Section 4 deals with the participation constraints. Here, these constraints involve additional difficulties, not present in an auction without externalities. In one dimensional mechanism design problems, the participation constraints bind only for the "lowest type". But, as the example in Rochet and Choné [16] suggests, this need not be the case in multidimensional problems. Moreover, in our environment, the buyers' reservation values are endogenous, and participation constraints are type-dependent. Thus, we must also construct optimal "threats" that ensure the participation of the buyers. Despite the relatively complex structure, we are able to show that it is enough to check the participation constraint for a "critical" type of every player.

In the rest of the paper we restrict attention to *symmetric settings* and to a class of anonymous mechanisms, which we call *standard bidding procedures*, for which we are able to deal with the integrability constraints. In a standard bidding procedure, agents can only submit one-dimensional "bids". Thus, buyers are not able to fully convey their types to the seller.

In Section 5 we confine attention to mechanisms that always transfer the object. In this case, we show that the optimal standard bidding procedure consists of a second-price auction. In the equilibrium of the second-price auction an agent makes a bid equal to the difference between his valuation for the object (net of externalities) and the expected externality suffered in case another agent gets the good.

In Section 6 we study second-price auctions with reserve prices (so the seller may keep the object sometimes). The integrability constraint plays a complex role when we try to determine the set of types that bid below the

reserve price (and hence never get the good). This constraint translates into a geometric condition about the direction of the normal to the boundary of the set of types M_0 that bid below the reserve price. The geometric condition can be stated as a differential equation, whose solution allows us to obtain the boundary of M_0, and explicitly construct the equilibrium strategies and the associated seller's revenue.

An interesting question is whether the seller will, independently of the relation between his characteristics and the buyers' characteristics, prefer to impose a strictly positive reserve price (i.e., will always exclude some types from consumption). The fact that exclusion is always optimal is a main finding in Armstrong's work. Contrary to Armstrong, we show by example that non-exclusion (i.e., imposing a reserve price equal to zero) is optimal for a range of the model's parameters.

2. THE MODEL

There are N buyers, indexed by $i = 1, ..., N$, and a seller, designated as player $i = 0$. We will refer to the "players" when we want to include the seller, and to the "buyers" when we want to exclude her, although the seller is not a player of the auction game. Let $I := \{1, ..., N\}$ be the set of buyers. The seller owns a single unit of an indivisible object.

Buyer i's preferences are determined by his vector type $t^i = (t^i_1, ..., t^i_N)$ where t^i_j is buyer i's payoff when player j gets the object.[1] A common situation, which we call the *negative externalities case*, is when each buyer receives a *positive* payoff if he obtains the object, and a *negative* payoff if anybody else gets it. But, in general, we will also allow for the possibility of positive externalities.

The seller has an analogous type represented by the N-dimensional vector t^0, which is common knowledge amongst the buyers (ex-ante). Buyer i's type is drawn from $T_i := (e_0, e_1]^{i-1} \times [v_0, v_1) \times (e_0, e_1]^{N-i}$, according with the density f_i, and is independent of all other players' types. Thus, the probability that the buyers' types are given by the N-tuple $(t^1, ..., t^N) \in T := T_1 \times \cdots \times T_N$ is $f(t) := f_1(t^1) \times \cdots \times f_N(t^N)$. Buyers' types are private information. The negative externalities case corresponds to $e_1 \leqslant 0$ and $v_0 \geqslant 0$. We allow for the possibility that $e_0 = -\infty$ and/or $v_1 = +\infty$. We assume that $f_i(t^i) > 0$ for all $t^i \in \mathrm{int}(T_i)$. The "origin" (or

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[1] We call this the *known default valued* model. Alternatively, we could also study an *unknown default values* model, where $t^i = (t^i_0, t^i_1, ..., t^i_n)$ and t^i_0 is buyer i's payoff when the seller keeps the object. The known default value model we study assumes that the values t^i_0, $i \in I$, are common knowledge and have been normalized to 0. The two models are similar and most of the characterization results we present apply equally to the unknown default values model (with small modifications).

"upper-left corner") of T_i is $C^i := (e_1, ..., e_1, v_0, e_1, ..., e_1)$; we also say that $t^i = C^i$ is player i's "lowest type".

Implicitly, we have made above a symmetry assumption that later plays an important role in the analysis: the extreme externalities e_0 and e_1 for a player i are *independent* of which other player gets the object. We have also assumed that appropriately permuting the coordinates, the origin C^i is the "same" for each player i; this, however, is irrelevant and assumed only for convenience.

Buyer i's utility is additively separable: if he pays x_i to the seller and player j gets the object, his utility is $t_j^i - x_i$, where we define $t_0^i := 0$. In general, a buyer's payment need not be zero even if he doesn't get the object. The seller's utility is also additively separable: if buyer i pays her x_i, $i \in I$, and she gives the object to player j (j could be 0), her utility is $t_j^0 + x_1 + \cdots + x_N$ (again, we define $t_0^0 := 0$).

For the study of general properties of auction mechanisms, there is no loss of generality in restricting attention to direct revelation mechanisms for which it is a Bayesian equilibrium for each buyer to report his type truthfully. Since a buyer cannot be forced to "participate" in the auction, "nonparticipation" must be included among his possible reports. Let $\Sigma := \{\sigma \in \mathbf{R}_+^N \mid \sum \sigma_i \leqslant 1\}$ be the set of *probability vectors*. The coordinate σ_i of a probability vector σ represents the probability that player i gets the object, and $\sigma_0 := 1 - \sum_{i \in I} \sigma_i$ represents the probability that the seller keeps the object. The seller specifies the rules of the auction in terms of a *revelation mechanism* (ρ, x, p), where $\rho = (\rho^1, ..., \rho^N) \in \Sigma^N$ is a profile of N probability vectors, $x_i : T \to \mathbf{R}$, $i \in I$, and $p : T \to \Sigma$. The seller asks each of the buyers simultaneously to report a type. If all buyers submit a type and the report profile is $(t^1, ..., t^N) \in T$, buyer i must pay the seller $x_i(t^1, ..., t^N)$, and he gets the object with probability $p_i(t^1, ..., t^N)$. If buyer i refuses to participate in the auction while all other buyers submit a report, the object is given to player j with probability ρ_j^i, $j \in I$, and no buyer makes a payment to the seller.[2] If two or more buyers refuse to submit a report, then, say, the seller keeps the object with probability 1 and nobody makes any payments.[3]

Suppose player i believes everybody else reports truthfully. Then, to assess the expected value of his reports, he only needs to know the conditional expected value, given his own type, of his payment and the probability

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[2] The probability vector ρ^i is designed to "punish" buyer i when he refuses to report his type. We assume that buyer i cannot be forced to accept the object, hence ρ_i^i is always 0.

[3] We study the Nash equilibria of the game, and disregard the possibility of coalition formation. Hence, profitable multiple deviations are irrelevant. If the domain of the function p were extended to include profiles where some players report "nonparticipation", then the vectors ρ^i could be included as part of the definition of p. We have chosen the domain T mainly to simplify the notation below; in an honest equilibrium, no player reports "nonparticipation".

assignment vector. Define then the functions $y_i: T_i \to \mathbf{R}$ and $q^i: T_i \to \Sigma$ as follows:

$$y_i(t^i) := \int_{T_{-i}} x_i(t^1, ..., t^N)\, f_{-i}(t^{-i})\, dt^{-i}$$

$$q^i_j(t^i) := \int_{T_{-i}} p_j(t^1, ..., t^N)\, f_{-i}(t^{-i})\, dt^{-i}.$$

We will refer to these functions as buyer i's *conditional expected payment* and *conditional expected probability assignment* in the mechanism (ρ, x, p). If buyer i believes his opponents will report truthfully, and reports type s^i when his type is t^i, his expected utility is $U_i(s^i, t^i) := q^i(s^i) \cdot t^i - y_i(s^i)$.

The auction mechanism (ρ, x, p) is said to be *incentive compatible* for buyer i if

$$U_i(t^i, t^i) \geqslant U_i(s^i, t^i) \qquad \text{for all} \quad s^i, t^i \in T_i,$$

and to satisfy the *participation constraint* for buyer i if

$$U_i(t^i, t^i) \geqslant \rho^i \cdot t^i \qquad \text{for all} \quad t^i \in T_i.$$

The right hand side of the last inequality is buyer i's expected value when he doesn't make any payments to the seller, and the seller assigns the object randomly according with the probability vector ρ^i. The auction mechanism is *feasible* if it is incentive compatible and satisfies the participation constraints for every buyer.

Clearly, buyer i cares only about his expected payment, and if (ρ, x, p) is a feasible mechanism, so is $(\rho, \bar{x}, p)$, where $\bar{x}_i(t) := y_i(t^i)$ for all $t \in T$. Moreover, the seller expects the same revenue with $\bar{x}$ or with x. Thus, there is no loss of generality in restricting attention to mechanisms for which the payment of each buyer depends only on his own report. Consequently, we will specify below auction mechanisms directly in terms of (ρ, y, p). However, for comparing the mechanism with traditional auctions, it is sometimes more convenient to describe explicitly how buyer i's payment changes with his opponents' bids (i.e., in a second-price auction).

3. INCENTIVE COMPATIBILITY

Any auction mechanism (ρ, y, p) presents each buyer i with a "menu"

$$M_i := \{(q^i(t^i), y_i(t^i)) \mid t^i \in T_i\}.$$

Buyer i's *surplus function* $S_i: T_i \to \mathbf{R}$ is then

$$S_i(t^i) := \sup \left\{ q^i(s^i) \cdot t^i - y_i(s^i) \mid s^i \in T_i \right\} \qquad t^i \in T_i.$$

This optimization problem determines buyers i's optimal report when he is presented with the menu M_i and his type is t^i. Various properties of S_i are readily available and have been previously recorded by Armstrong [1] and Rochet [15] (for a general reference on conjugate duality and properties of convex functions, the reader should consult the classic references by Rockafellar [17, 18]). S_i is convex, continuous, and monotonically increasing. Let $\partial S_i(t^i)$ denote the sub differential of S_i at t^i. Then, the following statements are equivalent:

(1) (ρ, y, p) is incentive compatible for buyer i;

(2) $S_i(t^i) = U_i(t^i, t^i)$ for all $t^i \in T_i$; and

(3) for all $t^i \in T_i$, $q^i(t^i) \in \partial S_i(t^i)$ and $y_i(t^i) = q^i(t^i) \cdot t^i - S_i(t^i)$.

Moreover, since S_i is convex, S_i is differentiable almost everywhere (a.e.), and if S_i is differentiable at t^i, $\partial S_i(t^i) = \{\nabla S_i(t^i)\}$. Thus $q^i(t^i) = \nabla S_i(t^i)$ a.e. in T_i.

Note that $S_i(t^i)$ is non decreasing in each of its arguments because $q^i(t^i)$, being a probability vector, is always nonnegative. The standard extension for vector-valued functions of the familiar notion of monotonicity for real-valued functions is given in the next definition. A function is convex only if its sub differential is a monotone map (Rockafellar [17]). Thus, if (ρ, y, p) is incentive compatible, q^i must be monotone for each i.

DEFINITION.   The function q^i is *monotone* if for every $s^i, t^i \in T_i$,

$$(s^i - t^i) \cdot (q^i(s^i) - (q^i(t^i)) \geqslant 0.$$

The property that $q^i(t^i) \in \partial S_i(t^i)$ for every $t^i \in T_i$ is the familiar "envelope condition". Let (ρ, y, p) be an incentive compatible auction mechanism. If S_i is differentiable at t^i, we have

$$\frac{dS_i}{dt_j^i}(t^i) = \frac{dU_i}{dt_j^i}(t^i, t^i) = \left.\frac{\partial U_i}{\partial t_j^i}(s^i, t^i)\right|_{s^i = t^i} = q_j^i(t^i),$$

because $\partial U_i(s^i, t^i)/\partial s_k^i|_{s^i = t^i} = 0$ for all k.

Since S_i is convex and $q^i(t^i) \in \partial S_i(t^i)$ for all t^i,

$$S_i(t^i) = S_i(s^i) + \int_{s^i}^{t^i} q^i(\tau) \cdot d\tau \qquad \text{for all} \quad s^i, t^i \in T_i.$$

The integral in the right hand side is a line integral, which does not depend on the specific path from s^i to t^i.[4] That is, the vector field q^i must be *conservative*.[5] The following proposition summarizes these results.

PROPOSITION 1. *Consider the auction mechanism (ρ, y, p), and let q^i, $i \in I$, be the corresponding conditional probability assignment functions. Then,*

(1) *The mechanism is incentive compatible for buyer i iff the vector field q^i: $T_i \to \Sigma$ is monotone and conservative, and for each $t^i \in T_i$,*

$$y_i(t^i) = q^i(t^i) \cdot t^i - S_i(t^i), \qquad where$$

$$S_i(t^i) = S_i(C^i) + \int_{C^i}^{t^i} q^i(s^i) \cdot ds^i.$$

(2) *the mechanism satisfies the participation constraints for buyer i iff*

$$S_i(t^i) \geqslant \rho^i \cdot t^i \qquad for\ all \quad t^i \in T_i.$$

By part (1), the payment and surplus of the lowest type, $y_i(C^i)$ and $S_i(C^i)$, satisfy $S_i(C^i) = q^i(C^i) \cdot C^i - y_i(C^i)$ (that is, the constant of integration $S_i(C^i)$ is uniquely determined by $y_i(C^i)$). Therefore, part (1) of Proposition 1 states that for incentive compatible mechanisms, the expected payment function y_i is uniquely determined by $y_i(C^i)$ and the probability assignment function p. We will denote the vector $(y_1(C^1), ..., y_N(C^N))$ by $y(C)$.

We next present a technical result, which provides a geometric characterization of the integrability condition for a class of piecewise continuous conditional expected probability assignment functions. Many traditional auctions, adapted to the current model with externalities, yield conditional expected probability assignment functions that fall within this class. More generally, this is often the case for deterministic mechanisms (where the rules of the auction specify that a.e. the object is assigned to a specific player with probability 1). Although we state the result for probability assignment functions, the proposition should prove useful in other contexts

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[4] If S_i is continuously differentiable in T_i, the above identity is a consequence of the fundamental theorem of calculus. The identity is also valid when S_i is a convex function, even if S_i is not differentiable everywhere. This has been shown by Krishna and Maenner [4]. Our Proposition 2 below develops the appropriate test for checking whether q^i is conservative, when q^i is discontinuous (i.e., when S^i is not differentiable) in a finite collection of manifolds.

[5] A vector field (or function) $v: \mathbf{R}^N \to \mathbf{R}^N$ is *conservative* if it is the gradient of a function $V: \mathbf{R}^N \to \mathbf{R}$. If v is differentiable, v is conservative iff $\partial v_i / \partial \tau_j \equiv \partial v_j / \partial \tau_i$ for all $i \neq j$. That is, iff the Jacobian of v is symmetric.

as well (i.e., nonlinear pricing). We use it in Section 6 to find the equilibrium of second-price auctions with reserve prices. The proof of Proposition 2 is relegated to the Appendix.

PROPOSITION 2. *Assume $q^i: T_i \to \Sigma$ is piecewise continuous. That is, assume there exists a partition $\{M_1, ..., M_k\}$ of T_i such that q^i restricted to $int(M_j)$ is continuous for each $j = 1, ..., k$. Suppose each M_j is an N-dimensional manifold with a piecewise smooth boundary*[6] *Then, q^i is conservative iff* (1) *q^i restrict to M_j is conservative for each $j = 1, ..., k$; and* (2) *whenever M_j and M_m are two adjacent regions, the jump in $q^i(t^i)$ as t^i crosses from M_j to M_m is perpendicular to the common boundary between M_j and M_m. That is, if t^i is in the common boundary between M_j and M_m, and n is the unitary normal vector of the boundary between M_j and M_m at t^i, then the vector*

$$\Delta q^i(t^i) := \lim_{\varepsilon \to 0^+} q^i(t^i + \varepsilon n) - \lim_{\varepsilon \to 0^+} q^i(t^i - \varepsilon n)$$

is parallel to n.

4. THE PARTICIPATION CONSTRAINTS

In this section we show that it is enough to verify the participation constraint for the critical type O^i, where O^i is the type in T_i closest to the point $(0, ..., 0, v_0, 0, ..., 0)$. If type O^i satisfies the participation constraint and the auction mechanism is incentive compatible, then *all* the other participation constraints are satisfied (and thus, they are redundant). Moreover, although it is possible to choose different critical types, from the seller's point of view, the choice of O^i above is *optimal*.

Let $p: T \to \Sigma$ be the probability assignment vector of an arbitrary incentive compatible mechanism. That is, $q^i: T_i \to \Sigma$ is monotone and conservative for each $i \in I$. We want to find $y: T \to \mathbf{R}^N$ and $\rho \in \Sigma^N$ such that (ρ, y, p) is feasible and maximizes the seller's expected revenue (for the given p).

By Proposition 1, y is completely determined by $y(C)$ and p (through the incentive compatibility constraints). For each $i \in I$, let

$$S_i^*(t^i) = q^i(C^i) \cdot C^i + \int_{C^i}^{t^i} q^i(s^i) \cdot ds^i, \qquad t^i \in T_i,$$

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[6] This is a slight abuse of terminology. We assume that each M_j contains its boundary, and therefore if M_j and M_m are two adjacent regions, $M_j \cap M_m$ is the common boundary between M_j and M_m, and therefore is not empty. Hence $\{M_1, ..., M_k\}$ is not quite a partition of T_i.

so that $S_i(t^i) = S_i^*(t^i) - y_i(C^i)$, $t^i \in T_i$. Then the problem is to find for each $i \in I$, the optimal solution $(\rho^i, y_i(C^i)) \in \Sigma \times \mathbf{R}$ of the problem

$$\max_{(\rho^i, y_i(C^i))} y_i(C^i)$$

$$\text{s.t. } S_i^*(t^i) - y_i(C^i) \geqslant \rho^i \cdot t^i \qquad \text{for all} \quad t^i \in T_i$$

$$\rho_i^i = 0.$$

Geometrically, we are choosing the slope (ρ^i) of a hyperplane through the origin of $\mathbf{R}^N$ and a downward vertical translation $(y_i(C^i))$ for the graph of S_i so that the translation is maximized subject to the constraint that the graph of S_i stay (weakly) above the hyperplane. Obviously, since S_i is convex, an optimal $(\rho^i, y_i(C^i))$ will make the hyperplane tangent to the graph of S_i at some critical point O^i, and therefore $\rho^i \in \partial S_i(O^i)$ (where we extend the domain of S_i to $\mathbf{R}^N$ by defining $S_i(t^i) = +\infty$ for all $t^i \notin T_i$).

We distinguish three cases (i) the negative externalities case when $e_1 < 0$, (ii) the positive externalities case when $e_0 > 0$, and (iii) the mixed case when $e_0 \leqslant 0 \leqslant e_1$. If $e_1 < 0$, all the externalities are negative, and the most severe threat the seller can make is to give the object to one of i's opponents for sure if i does not participate. That is, she should pick $\rho^i \in \Sigma$ such that $\rho_i^i = 0$ and $\sum_{j \in I} \rho_j^i = 1$. Since the subgradient $q^i(t^i)$ of S_i at t^i remains in Σ, if $y_i(C^i)$ is increased until $S_i(C^i) = \rho^i \cdot C^i \equiv e_1$, the graph of S_i will be above the hyperplane everywhere else. That is, the critical type is $O^i = C^i$. If $e_0 > 0$, then all the externalities are positive and the seller should commit to keeping the object when buyer i does not participate. That is, she should choose $\rho^i = 0$ (a horizontal hyperplane). Since $q^i(t^i) \geqslant 0$ for all t^i, the function S_i is increasing and the critical type is the point $O^i = (e_0, ..., e_0, v_0, e_0, ..., e_0)$, where S_i attains its minimum. In this case, $y_i(C^i)$ can be increased until $S_i(O^i) = 0$. Finally, when $e_0 \leqslant 0 \leqslant e_1$, the externalities are mixed. Here the critical type is $O^i = (0, ..., 0, v_0, 0, ..., 0)$. To see this, recall that $\rho_i^i = 0$ and therefore the hyperplane must contain the point $(O^i, 0)$. Since the graph of S_i must remain above the hyperplane, we must have that $S_i(O^i) \geqslant 0$. It turns out that we can increase $y_i(C^i)$ until $S_i(O^i) = 0$ provided we pick $\rho^i \in \partial S_i(O^i)$. We have that $q^i(O^i) \in \partial S_i(O^i)$, and since $\rho_i^i = 0$, in general $\rho^i \neq q^i(O^i)$. Yet, to choose $\rho^i \in \partial S_i(O^i)$ is possible because O^i is on the boundary of T_i, and hence $\partial S_i(O^i)$ is not a singleton.

The above optimal translations $y_i(C^i)$ are

$$\hat{y}_i(C^i) = \begin{cases} q^i(C^i) \cdot C^i - e_1 & \text{in case (i)} \\ q^i(C^i) \cdot C^i + \displaystyle\int_{C^i}^{O^i} q^i(s) \cdot ds & \text{in case (ii) and (iii),} \end{cases}$$

and the corresponding optimal threat vectors are

case (i): any vector in $\hat{\rho}^i \in \Sigma$ such that $\hat{\rho}_i^i = 0$, $\hat{\rho}_j^i \geq q_j^i(C^i)$ for all $j \neq i$, and $\sum_{j \in I} \hat{\rho}_j^i = 1$;

case (ii): $\hat{\rho}^i = 0$;

case (iii): $\hat{\rho}^i = (\hat{\rho}_i^i, \hat{\rho}_{-i}^i) = (0, q_{-i}^i(O^i))$ (where $O^i = (0, ..., 0, v_0, 0, ..., 0)$).

The following proposition summarizes our discussion. Its proof in the Appendix verifies that $\hat{\rho}^i$ is the appropriate slope for the hyperplane in each of the three cases.

PROPOSITION 3. *Let p be such that every q^i, $i \in I$, is monotone and conservative. Then $(\hat{\rho}, \hat{y}(C), p)$ is the optimal auction mechanism among all auction mechanisms with probability assignment function p. In particular, if $(\rho, y(C), p)$ is a feasible mechanism (not necessarily optimal), then $y(C) \leq \hat{y}(C)$ and $(\hat{\rho}, y(C), p)$ is also feasible.*

Proposition 3 shows that as in Myerson's auction model, it is enough to check the participation constraint for only one critical type. Here, buyer i's critical type is O^i, which the closest type to $(0, ..., v_0, ..., 0)$.

5. STANDARD BIDDING MECHANISMS FOR THE SYMMETRIC CASE

We now study more traditional auctions in which each buyer i submits a *one-dimensional* bid b_i. For symmetric settings, we derive the equilibrium of the second-price auction with an entry fee. We then show that this auction maximizes the seller's expected revenue among all standard auctions (with one-dimensional bids) where the seller never keeps the good. In particular, in this framework we can deal with the complex integrability constraints (that each q^i be conservative). Finally we note that the second-price auction with entry fee is optimal in the class of *all* auctions that always transfer the good if there are only two buyers, or if there are n buyers, but for all $j \neq i \neq k$, $t_j^i = t_k^i$.

We will now assume that the set of types for each buyer is bounded. Hence, $T_i = [e_0, e_1]^{i-1} \times [v_0, v_1] \times [e_0, e_1]^{N-i}$, where $v_0 > 0$, v_1, e_0 and e_1 are all finite real numbers.

In a *standard bidding mechanism*, the rules of the auction are specified by the probability assignment function $p: [\underline{b}, \bar{b}]^N \to \Sigma$, the payment functions $x_i: [\underline{b}, \bar{b}]^N \to \mathbf{R}$, $i \in I$, and $\rho = (\rho^1, ..., \rho^N)$, where $x_i(b_i, ..., b_N)$ and $p_i(b_1, ..., b_N)$ are, respectively, player i's payment and probability of winning the object given the bid profile $b = (b_1, ..., b_N)$, and ρ^i is the lottery used to assign the object if buyer i does not participate. The latter is chosen according

with Proposition 3, and is not mentioned again in the analysis below. The interval $[\underline{b}, \bar{b}]$ represents the set of *admissible bids*; an inadmissible bid is equivalent to nonparticipation.

In this section, all mechanisms are assumed to be standard, unless otherwise specified.

Situations in which the buyers are ex-ante identical are relatively natural. A typical requirement in the model without externalities is that the buyers' types are all drawn from the same distribution. Here we require more. In particular, if i, j, k are three different buyers, and τ_1 and τ_2 are two possible externalities, we assume that it is equally likely that $(t_j^i, t_k^i) = (\tau_1, \tau_2)$ or $(t_j^i, t_k^i) = (\tau_2, \tau_1)$.

A *permutation* of I is any bijection $\pi: I \to I$. A permutation π *fixes* i if $\pi(i) = i$. For each permutation π of I, we define the function $\Pi: \mathbf{R}^N \to \mathbf{R}^N$ as follows: $\Pi(b) := (b_{\pi^{-1}(1)}, ..., b_{\pi^{-1}(N)})$ for each $b \in \mathbf{R}^N$. Let π_{ij} be the *simple permutation* that switches the indices i and j (that is, $\pi_{ij}(i) = j$, $\pi_{ij}(j) = i$, and $\pi_{ij}(k) = k$ for all $k \notin \{i, j\}$), and let $\Pi_{ij}: \mathbf{R}^N \to \mathbf{R}^N$ be the corresponding map switching coordinates i and j.

DEFINITION. The setting is *symmetric* if

(S_1) $t_i^0 = t_j^0$ for all $i, j \in I$.

(S_2) for each $i \in I$, type t^i, and permutation π that fixes i, $f_i(\Pi(t^i)) = f_i(t^i)$.

(S_3) for any $i \neq j$ and type t^i, $f_i(t^i) = f_j(\Pi_{ij}(t^i))$.

Condition (S_1) says that no matter which buyer gets the object, the seller suffers the same externality. (S_2) states that the probabilities that buyer i is of type t^i or another type equal to t^i but with coordinates j and k (where $j \neq i$ and $k \neq i$) switched, are the same. Finally, (S_3) says that types for each player are drawn from the "same" distribution. For the rest of this section, we assume that the setting is always symmetric.

We restrict attention to *anonymous* mechanisms in which the seller cannot make the outcome depend on the identity of the buyers,[7] and to symmetric equilibria, where all buyers use the same bidding strategy.

DEFINITION. The mechanism (p, x) is *anonymous* if for any bid profile $b \in [\underline{b}, \bar{b}]^N$,

(A_1) $p_i(\Pi(b)) = p_i(b)$ and $x_i(\Pi(b)) = x_i(b)$ for any $i \in I$ and permutation π that fixes i; and

(A_2) $p_j(\Pi_{ij}(b)) = p_i(b)$ and $x_j(\Pi_{ij}(b)) = x_i(b)$ for any $i, j \in I$ with $i \neq j$.

[7] For example, we imagine that each buyer has been assigned a secret number that he uses to identify his bid. However, the seller does not observe which number is assigned to each buyer, and thus does not know which buyer submits what bid.

In an anonymous mechanism, buyer i's payment and probability of getting the object are unaffected if two of his opponents swap their bids. And, if buyers i and j swap their bids, their corresponding payments and probabilities of getting the object are swapped too. (A_1) and (A_2) together imply that $p_j(\Pi(b)) = p_i(b)$ and $x_j(\Pi(b)) = x_i(b)$ for any $i \neq j$ and permutation π such that $\pi(i) = j$.

DEFINITION. A bidding strategy for player i is a function $B_i \colon T_i \to [\underline{b}, \bar{b}]$. A bidding strategy profile $B = (B_1, \ldots, B_N)$ is *symmetric* if $B_i(t^i) = B_1(\Pi_{1i}(t^i))$ for all $t^i \in T_i$ and $i \in I$.[8]

Given a standard bidding mechanism (p, x) and a bidding strategy profile $B = (B_1, \ldots, B_N)$, player i's conditional payment $y_i(b_i)$ and probability assignment vector $q^i(b_i)$ for any bid $b_i \in [\underline{b}, \bar{b}]$ are defined by

$$y_i(b_i) := \int_{T_{-i}} x_i(b_i, B_{-i}(t^{-i}))\, f_{-i}(t^{-i})\, dt^{-i}$$

$$q^i_j(b_i) := \int_{T_{-i}} p_j(b_i, B_{-i}(t^{-i}))\, f_{-i}(t^{-i})\, dt^{-i}, \qquad j \in I.$$

It is easy to see that if (p, x) is an anonymous mechanism and B is a symmetric bidding strategy, then $y_i \equiv y_1$ for all $i > 1$. That is, the expected payment function is the same for all buyers. The next lemma also establishes that the conditional probability assignment function is the same for all the buyers up to a permutation of coordinates. Moreover, for any buyer i and any bid b_i, each of i's opponents has the same conditional probability of getting the object. The proof of the lemma is relegated to the Appendix.

LEMMA 1. *Suppose (p, x) is anonymous and B is a symmetric bidding strategy. Then, for any $i \in I$, $\beta \in [\underline{b}, \bar{b}]$, and $j \in I \setminus \{i\}$,*

$$q^i_j(\beta) = \frac{1 - q^i_0(\beta) - q^i_i(\beta)}{N - 1} \qquad and \qquad q^i_i(\beta) = q^1_1(\beta).$$

Note that Lemma 1 also implies that for any $\beta \in [\underline{b}, \bar{b}]$ and $i \in I$, $q^i_{-0}(\beta) = \Pi_{1i}(q^1_{-0}(\beta))$.

Let B be a symmetric strategy for the anonymous mechanism (p, x). Buyer i's expected utility when he is of type $t^i \in T_i$ and bids $b_i \in [\underline{b}, \bar{b}]$ is

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[8] It may also seem natural to require in the definition of a symmetric bidding strategy that $B_1(\Pi_{ij}(t^1)) = B_1(t^1)$ for all $t^1 \in T_1$ and $i \neq 1 \neq j$. However, this condition is not needed for our results.

$$U_i(b_i, t^i) := q^i(b_i) \cdot t^i - y_i(b_i)$$

$$= q_i^i(b_i)\left[t_i^i - \frac{1}{N-1}\sum_{j\neq i} t_j^i \right] + (1 - q_0^i(b_i))\left[\frac{1}{N-1}\sum_{j\neq i} t_j^i \right] - y_i(b_i),$$

where the second equality follows from Lemma 1.

DEFINITION. Let $\Sigma_0 := \{\sigma \in \Sigma \mid \sum_{i \in I} \sigma_i = 1\}$. The mechanism (p, x) always transfers the object if $p(b) \in \Sigma_0$ for all $b \in [\underline{b}, \overline{b}]^N$ (that is, the probability that the seller keep the object is 0).

If B is a symmetric bidding strategy for a mechanism (p, x) that always transfers the object, then

$$U_i(b_i, t^i) = q_i^i(b_i)\left[t_i^i - \frac{1}{N-1}\sum_{j\neq i} t_j^i \right] + \frac{1}{N-1}\sum_{j\neq i} t_j^i - y_i(b_i),$$

because $q_0^i(b_i) = 0$ and $q_j^i(b_i) = (1 - q_i^i(b_i))/(N-1)$ for all $j \neq i$. This decomposition shows that given a symmetric bidding strategy, buyer i's expected payoff is determined completely by his bid and the difference between his valuation of the object and the average externality he suffers in case another buyer gets the object. Accordingly, we make the following definition.

DEFINITION. Let $\underline{b}^* := v_0 - e_1$ and $\overline{b}^* := v_1 - e_0$, and B^* be the symmetric bidding strategy with range $[\underline{b}^*, \overline{b}^*]^N$, where

$$B_i^*(t^i) = t_i^i - \frac{1}{N-1}\sum_{j\neq i} t_j^i, \qquad t^i \in T_i.$$

LEMMA 2. *Let (p, x) be an anonymous mechanism that always transfers the object. Suppose the symmetric bidding strategy B is an equilibrium for (p, y). Then, for all $i \in I$ and a.e. $t^i, s^i \in T_i$, $B_i^*(t^i) = B_i^*(s^i)$ implies $B_i(t^i) = B_i(s^i)$.*[9]

Proof. By symmetry, it is enough to show the result for $i = 1$. Player 1's surplus function is given by $S_1(t^1) := \sup \{U_1(b_1, t^1) \mid b_1 \in [\underline{b}, \overline{b}]\}$. Let $M := \{(q_1^1(b_1), y(b_1)) \mid b_1 \in [\underline{b}, \overline{b}]\}$ be the menu set.[10] The surplus function is convex, and b_1^* maximizes player 1's expected utility $U_1(b_1, t^1)$ (that is, $b_1^* = B_1(t^1)$) iff $q^1(b_1^*) \in \partial S_1(t^1)$.

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[9] Lemma 2 leaves open the possibility that for a set E of Lebesgue measure 0 in the interval $[\underline{b}^*, \overline{b}^*]$, there exist types t^1 and s^1 such that $B_1^*(t^1) = B_1^*(s^1) \in E$ and $B_1(t^1) \neq B_1(s^1)$. But then, we could redefine the equilibrium strategy as follows. For each $b_1 \in E$, pick an arbitrary t^1 with $B_1^*(t^1) = b_1$, and for all $s^1 \in T_1$ with $B_1^*(s^1) = b_1$, let $B_1(s^1) = B_1(t^1)$ (which is also optimal at s^1). Redefine B_i, $i \geqslant 2$, accordingly to maintain symmetry. This will not affect any of the players' expectations, and the new B will remain a symmetric equilibrium.

[10] By symmetry, every player i faces the same menu M, and therefore $q_i^i(b_i) = q_1^1(b_i)$ for all i and $b_i \in [\underline{b}, \overline{b}]$.

Let $t^1, s^1 \in T_1$ be such that $B_1^*(t^1) = B_1^*(s^1)$. Since $p_0(b) = 0$ for all $b \in [\underline{b}, \bar{b}]^N$, $q_0^1(b_1) = 0$ for all $b_1 \in [\underline{b}, \bar{b}]$, and

$$U_1(b_1, t^1) = U_1(b_1, s^1) + \frac{1}{N-1} \sum_{j \neq 1} [t_j^1 - s_j^1].$$

Therefore, $S_1(t^1) = S_1(s^1) + (1/N-1) \sum_{j \neq 1} [t_j^1 - s_j^1]$, and $\partial S_1(t^1) = \partial S_1(s^1)$. Moreover, S_1 is differentiable almost everywhere, and S_1 is differentiable at t^1 iff S_1 differentiable at s^1. If S_1 is differentiable at t^1 (and s^1), then

$$q_1^1(B_1(t^1)) = \frac{\partial S_1}{\partial t_1^1}(t^1) = \frac{\partial S_1}{\partial t_1^1}(s^1) = q_1^1(B_1(s^1)),$$

and thus, by the monotonicity of q^1, $B_1(t^1) = B_1(s^1)$. ∎

A symmetric revelation principle. By Lemma 2, for any anonymous mechanism (p, x) that always transfers the object and corresponding symmetric equilibrium B in which all types of every player participate, there exists another anonymous mechanism $(\tilde{p}, \tilde{x})$ for which B^* is an equilibrium, and such that for each i and $t^i \in T_i$,

$$q_i^i(B_i(t^i)) = \tilde{q}_i^i(B_i^*(t^i)) \qquad \text{and} \qquad y_i(B_i(t^i)) = \tilde{y}_i(B_i^*(t^i)).$$

Here $q_i^i(t^i)$ and $\tilde{q}_i^i(t^i)$ (y_i and $\tilde{y}_i$) are respectively the conditional probabilities of winning the object (payments) in the equilibria B and B^*. Indeed, for any $b \in [\underline{b}^*, \bar{b}^*]^N$, choose any $t \in T$ such that $b = (B_1^*(t^1), ..., B_N^*(t^N))$, and let

$$\tilde{p}_i(b) := p_i(B_1(t^1), ..., B_N(t^N)) \qquad \text{and} \qquad \tilde{y}_i(b_i) := y_i(B_i(t^i)), \qquad i \in I.$$

That is, without loss of generality, for symmetric equilibria of anonymous mechanisms that always transfer the object, we can restrict attention to direct revelation mechanisms in which each player i reveals his "summary type" $B_i^*(t^i)$.

We can also view any anonymous mechanism (p, x) for which B^* is an equilibrium as a direct revelation mechanism $(\hat{p}, \hat{x})$ for which truthtelling (and participation) is an equilibrium. We can define $(\hat{p}, \hat{x})$ as follows:

$$\hat{p}(t) := p(B_1^*(t^1), ..., B_N^*(t^N)) \qquad \text{and}$$
$$\hat{x}(t) := x(B_1^*(t^1), ..., B_N^*(t^N)), \qquad t \in T.$$

For each i and $t^i \in T_i$, consider the following change of variables:

$$s_i^i := B_i^*(t^i), \qquad \text{and} \qquad s_j^i := t_j^i \quad \text{for all } j \neq i.$$

The determinant of the Jacobian of this change of variables is 1, and its inverse is $t^1 = (\tau_i(s^i), s^i_{-i})$, where

$$\tau_i(s^i) := s^i_i + \frac{1}{N-1} \sum_{j \neq i} s^i_j.$$

For each $i \in I$, $s^i_i \in [\underline{b}^*, \bar{b}^*]$, and $s^i_{-i} \in [e_0, e_1]^{N-1}$, define

$$\hat{f}_i(s^i) := f_i(\tau_i(s^i), s^i_{-i}),$$

$$g_i(s^i_i) := \int \hat{f}_i(s^i) \, ds^i_{-i}, \qquad G_i(s^i_i) := \int_{\underline{b}^*}^{s^i_i} g_i(z) \, dz, \qquad \text{and}$$

$$\gamma_i(s^i_i) := s^i_i - \frac{1 - G_i(s^i_i)}{g(s^i_i)}.$$

$g_i(s^i_i)$ represents the density of all types that have the same summary type s^i_i (and G_i is its corresponding distribution), and $\gamma_i(s^i_i)$ is the analogue of Myerson's virtual type. The symmetry assumptions (S_2) and (S_3) imply that $g_i \equiv g_1$ and $\gamma_1 \equiv \gamma_1$ for all $i \in I$. Hence, hereafter we drop the subindex from these two functions.

Consider now the seller's problem of finding the standard mechanism that maximizes expected revenue. By explicitly carrying out the integration with respect to s^i_{-i} for each i, the seller's problem can be stated as

$$(P) \qquad \max \int_{[\underline{b}^*, \bar{b}^*]^N} \sum_{i \in I} [\gamma(b_i) \cdot p_i(b_1, ..., b_N)] \, g(b_1) \cdots g(b_N) \, db_1 \cdots db_N$$

$$\text{s.t. } p: [\underline{b}^*, \bar{b}^*]^N \to \Sigma, \qquad \text{and}$$

monotonicity and integrability constraints.

We now identify a class of problems for which it is possible to relax both the monotonicity *and* integrability constraints. If the problem is "regular", then the solution of the relaxed problem satisfies these constraints automatically.

DEFINITION. Problem (P) is *regular* if $\gamma(b) = b - [1 - G(b)]/g(b)$ is an increasing function of $b \in [\underline{b}^*, \bar{b}^*]$.

The second-price auction. Let $[\underline{b}^*, \bar{b}^*]^N$ be the set of admissible bids. If player i does not participate or submits an invalid bid, we make his bid $b_i = *$. Let J denote the set of all bidders who submit a valid bid, W be

the subset of those who submit the largest (valid) bid, and $\#(J)$ and $\#(W)$ be their corresponding cardinalities. Pick any $i \in W$, and let

$$X := \begin{cases} v_0 - e_1 & \text{if } \#(J) = 1 \\ \max\{b_j \mid j \in J \setminus \{i\}\} & \text{if } \#(J) \geqslant 2. \end{cases}$$

Then, for each $i \in I$, buyer i's payment and probability of getting the object are given by

$$(x_i^s(b_1, ..., b_N), p_i^s(b_1, ..., b_N)) = \begin{cases} \left(\dfrac{X}{\#(W)}, \dfrac{1}{\#(W)} \right) & \text{if } i \in W \\ (0, 0) & \text{otherwise.} \end{cases}$$

Observe that we have extended the domain of the functions p_i and x_i to R^N, where $R := \{ * \} \cup [\underline{b}^*, \bar{b}^*]$, and thus the auction specifies an outcome for any subset of buyers who submit a valid bid. Obviously, the second-price auction is anonymous.

PROPOSITION 4. B^* *is the unique symmetric equilibrium of the second-price auctions.*

Proof. The proof follows the standard argument that shows that in the case without externalities, bidding one own's valuation is a dominant strategy. By symmetry, it is enough to show that for any t^1, $B_1^*(t^1)$ is a best response for player 1 to B_{-1}^*. Let $t^1 \in T_1$ and $b_1 := B^*(t^1)$. Consider another bid $b_1' < b_1$. Let h denote the highest bid submitted by 1's opponents. If $h < b_1' < b_1$, then player 1 gets the object whether he bids b_1 or b_1', and pays h in both cases. Thus, in both cases his payoff is $t_1^1 - h$. If $b_1' < b_1 < h$, then player 1 does not get the object when he bids b_1 or b_1', and his payoff is t_j^1 in both cases, where j is the winner of the object. However, if $b_1' < h < b_1$, player 1 gets the object when he bids b_1 but not when he bids b_1'. In the former case his payoff is $t_1^1 - h$, while in the latter, his (expected) payoff is $(1/N - 1) \times \sum_{j>1} t_j^1$. But, $h < b_1 = B^*(t^1)$ is equivalent to

$$t_1^1 - h > \frac{1}{N-1} \sum_{j>1} t_j^1,$$

and player 1 is (ex-ante) strictly better off when he bids b_1. Since the probability that the opponents' highest bid falls between b_1' and b_1 is strictly positive, $U_1(b_1, t^1) > U_1(b_1', t^1)$. The analysis showing that player 1 strictly prefers b_1 to a bid $b_1' > b_1$ is similar.

Finally, consider the bid $b_1' = *$. According with the rules of the auction, bidding $b_1' = *$ leads to exactly the same outcome as bidding $b_1' = \underline{b}^*$. ∎

Associated with the second-price auction (x^s, p^s), there is an equivalent direct revelation mechanism $(\hat{\rho}^s, \hat{x}^s, \hat{p}^s)$, defined as follows. For each $t \in T$,

$$\hat{p}^s(t^1, ..., t^N) := p^s(B_1^*(t^1), ..., B_N^*(t^N)),$$

$$\hat{x}^s(t^1, ..., t^N) := x^s(B_1^*(t^1), ..., B_N^*(t^N)),$$

and $\hat{\rho}^s = (\hat{\rho}^{s1}, ..., \hat{\rho}^{sN})$ is the corresponding profile of probability vectors defined in Proposition 3. Actually, the direct revelation mechanism $(\hat{\rho}^s, \hat{x}^s, \hat{p}^s)$ differs from the second-price auction (x^s, p^s) in that it makes a different threat for the case in which a buyer does not participate. In (x^s, p^s) the auction is run with as many buyers who decide to participate. Instead, in $(\hat{\rho}^s, \hat{x}^s, \hat{p}^s)$, the object is allocated according with the lottery ρ^{si} when buyer i does not participate. By Proposition 3, the seller's expected revenue can also be improved in $(\hat{\rho}^s, \hat{x}^s, \hat{p}^s)$ with an appropriate translation of the payments. Although in Proposition 3 we achieved this by increasing the payment of every type of every buyer, now we prefer to present this translation in terms of an entry fee. The entry fee is determined so as to extract the most surplus from the buyers while still ensuring that all types participate in the auction.

To avoid case (iii) in Proposition 3, which involves an endogenous computation, the rest of the section assumes that either case (i) or case (ii) holds.

The second-price auction with entry fee.   Each player must pay an entry fee $E = \max\{0, e_0\}$ to participate. If all buyers participate and submit a valid bid, the payments and probability assignment vector are given by (x^s, p^s). If buyer i does not participate or does not make a valid bid, the object is allocated according with the lottery $\hat{\rho}^{si}$, where

$$(\hat{\rho}_i^{si}, \hat{\rho}_{-i}^{si}) = \begin{cases} \dfrac{1}{N-1}(0, 1, ..., 1) & \text{if} \quad e_1 \leqslant 0 \\ (0, 0, ..., 0) & \text{if} \quad e_0 \geqslant 0. \end{cases}$$

The second-price auction with entry fee is a standard anonymous mechanism that always transfers the object (in equilibrium).

PROPOSITION 5.   *Suppose problem* (P) *is regular and that either* $(i)\, e_1 \leqslant 0$, *or* $(ii)\, e_0 \geqslant 0$. *Then the optimal standard anonymous mechanism that always transfers the object is the second-price auction with entry fee. In its corresponding* (*best*) *equilibrium, all buyers participate* (*pay the entry fee*), *and bid according with the strategy* B^*.

Proof.   If we require that $p: [\underline{b}^*, \bar{b}^*]^N \to \Sigma_0$ (instead of Σ), and drop the monotonicity and integrability constraints of problem (P), the relaxed

problem can be solved by pointwise maximization. The solution of the latter is

$$p(b_1, ..., b_N) \in \arg\max_{z \in \Sigma_0} \sum_{i=1}^{N} \gamma(b_i)\, z_i, \qquad \text{for each} \quad b \in [\underline{b}^*, \bar{b}^*]^N.$$

That is, the object is assigned (with probability 1) to the player i with the largest value of $\gamma(b_i)$. Since the problem is regular, this is the player with the largest bid b_i. Hence, the pointwise maximization yields the solution $p \equiv p^s$. By Proposition 4, the mechanism (x^s, p^s) has the unique symmetric equilibrium B^*. Hence, the direct revelation mechanism $(\hat{\rho}^s, (E, ..., E) + \hat{x}^s, \hat{p}^s)$ is feasible, and therefore, by Proposition 1, $\hat{p}^s$ is monotone and conservative.

The reader can check that $E + x_i^s$ implies the same optimal expected payment $\hat{y}_i(C^i)$ defined in Proposition 3 for the cases (i) and (ii). Hence, $(\hat{\rho}^s, (E, ..., E) + \hat{x}^s, \hat{p}^s)$ is the optimal direct revelation mechanism for the given $\hat{p}^s$.

Remark. As we argue below, when $N = 2$ and the seller always transfers the object, the restriction to standard mechanisms is not binding. Consider an incentive compatible auction mechanism (ρ, y, p) that always transfers the object ($p(t) \in \Sigma_0$ for all $t \in T$). If S_i denotes buyer i's surplus function, by Proposition 1 we have that $\nabla S_i(t^i) = q^i(t^i)$ a.e. $t^i \in T_i$, and therefore

$$\sum_{j \in I} D_j S_i(t^i) = 1,$$

where to avoid confusion below, we are denoting the partial derivative of S_i with respect to its jth argument at t^i by $D_j S_i(t^i)$. If $e = (1, 1)$, the above equality implies that $S_i(t^i + \alpha e) = S_i(t^i) + \alpha$ for all α in the real interval where $t^i + \alpha e$ remains in T_i.

Consider now two types $s^1, t^1 \in T_1$ such that $B_1^*(s^1) = B_1^*(t^1)$. That is, $s_1^1 - s_2^1 = t_1^1 - t_2^1$, or $s_1^1 - t_1^1 = s_2^1 - t_2^1$. Hence, we have that $s^1 = t^1 + (s_2^1 - t_2^1)e$, and thus $S_1(s^1) = S_1(t^1) + (s_2^1 - t_2^1)$. Moreover, for any $\varepsilon \in \mathbf{R}$ such that $(s_1^1 + \varepsilon, s_2^1), (t_1^1 + \varepsilon, t_2^1) \in T_1$, we also have that $S_1(s_1^1 + \varepsilon, s_2^1) = S_1(t_1^1 + \varepsilon, t_2^1) + (s_2^1 - t_2^1)$. This implies that

$$q_1^1(s^1) = D_1 S_1(s^1) = \lim_{\varepsilon \to 0} \frac{S_1(s_1^1 + \varepsilon, s_2^1) - S_1(s^1)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{S_1(t_1^1 + \varepsilon, t_2^1) - S_1(t^1)}{\varepsilon} = D_1 S_1(t^1) = q_1^1(t^1),$$

and that $q_2^1(s^1) = 1 - q_1^1(s^1) = 1 - q_1^1(t^1) = q_2^1(t^1)$. By incentive compatibility, we must also have that $y_1(s^1) = y_1(t^1)$. Define the standard auction mechanism $(\rho, \hat{y}, \hat{p})$ as follows: for each $(b_1, b_2) \in [\underline{b}^*, \bar{b}^*]^2$, pick an *arbitrary* $t \in T$ such that $B_i^*(t^i) = b_i$, $i \in I$, and let

$$\hat{p}(b_1, b_2) = p(t) \qquad \text{and} \qquad \hat{y}_i(b_i) = y_i(t^i), \qquad i \in I.$$

The argument above shows that B^* is an equilibrium of $(\rho, \hat{y}, \hat{p})$. Thus, when $N = 2$ and the seller always transfers the object, there is no loss of generality in restricting attention to standard bidding mechanisms. Therefore, when $N = 2$, by Proposition 5, the second-price auction (with appropriate entry fees) is optimal among *all* auctions that always transfer the object (standard or otherwise).

The same conclusion holds for $N \geqslant 3$, when the type spaces are two-dimensional in the following way.[11]

$$T_i = \{(t_i^i, t_{-i}^i) = (\alpha, \beta, ..., \beta) \mid \alpha \in [v_0, v_1] \text{ and } \beta \in [e_0, e_1]\}, \qquad i \in I.$$

A similar argument to that above shows that in this case too the second-price auction (with appropriate entry fees) is optimal among all auctions that always transfer the object.

6. SECOND-PRICE AUCTIONS WITH RESERVE PRICES

We now relax the constraint of the previous section, that the auction always transfers the object (in equilibrium), and investigate whether it is beneficial for the seller to keep the object sometimes. More specifically, we now consider a second-price auction with reserve price x_0: if each buyer's bid is below the reserve price x_0, the object remains with the seller; otherwise, the highest bidder gets the object and pays a price equal to the largest of the second highest bid and the reservation price x_0. The one-dimensional case suggests that reserve prices may enhance the seller's revenue, and we are interested in determining the optimal reserve price x_0.

Equilibrium. We construct a symmetric equilibrium $B = (B_1, ..., B_N)$ with the following characteristics. There is a convex region M_0 such that

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[11] Here, in contrast to our general setup, the dimensionality of the type spaces does not coincide with the number of players. Nevertheless, all our previous analysis extends to this case as well, because this coincidence is irrelevant.

player 1 bids his summary type $B_1^*(t^1)$ if $t^1 \notin M_0$, and bids below x_0 if $t^1 \in M_0$, where $B_1^*(t^1)$ is as defined in Section 5.[12]

As we argue more precisely below (for the case $N = 2$, although the analysis easily extends to the case $N > 2$ as well), bidding $B_1^*(t^1)$ for types $t^1 \notin M_0$ is optimal for very much the same reasons of Proposition 4. In order for B to constitute an equilibrium, the induced conditional probability assignment function q^1 must be conservative in T_1. Clearly, q^1 is conservative in M_0 and $T_1 \setminus M_0$, respectively. However, $q^1(t^1)$ is discontinuous as t^1 crosses the boundary between M_0 and $T_1 \setminus M_0$. Therefore, by Proposition 2, the vector field q^1 is conservative iff the jump in $q^1(t^1)$ is perpendicular to the boundary of M_0 as t^1 crosses that boundary. It turns out that this condition together with a boundary condition (see below) fully characterizes the region M_0. Thus, there is a unique candidate equilibrium B. In an example, we explicitly construct that B, and verify that it is an equilibrium. For this model, Proposition 2 proves to be a very powerful characterization result.

Formally, suppose the boundary of M_0 can be parametrized by a function $h: \mathbf{R}^{N-1} \to \mathbf{R}$. That is,

$$M_0 = \{(t_1^1, t_{-1}^1) \in T_1 \mid v_0 \leqslant t_1^1 \leqslant h(t_{-1}^1)\}.$$

Then, $t^1 \in T_1$ is in the boundary of M_0 iff $t_1^1 = h(t_{-1}^1)$, and vector normal to the boundary of M_0 at t^1 is

$$n = \begin{pmatrix} 1 \\ -\nabla h(t_{-1}^1) \end{pmatrix}.$$

Hence, by Proposition 2, the jump $\Delta q^1(t^1)$ must be parallel to n for all $t^1 \in T_1$ with $t_1^1 = h(t_{-1}^1)$. That is,

$$\Delta q_j^1(t^1) \bigg/ \frac{\partial h}{\partial t_j^1}(t_{-1}^1) = -\Delta q_1^1(t^1) \qquad \text{for each} \quad j > 1.$$

[12] Alternatively, we could have introduced entry fees instead of (or in addition to) reserve prices. However, one can view an entry fee as an equivalent reserve price, and vice-versa. For example, in the equilibrium just described, we could replace the reservation price x_0 by an entry fee $e = m_0^{N-1} x_0$, where m_0 is the probability that $t^1 \in M_0$. Then, the symmetric equilibrium B for the auction with reserve price corresponds to the following "equivalent" symmetric equilibrium for the auction with entry fees. Types $t^1 \in M_0$ do not participate, and types $t^1 \notin M_0$ pay the entry fee and bid $B_1^*(t^1)$, as before. The reader can check that in this equilibrium, the winner and each player's total expected payment is the same as in the equilibrium B of the auction with reserver price. A type $t^1 \notin M_0$, for example, now pays the entry fee e for sure. But, he also pays nothing when all his opponents' types are in (their corresponding) region M_0, while with reserve prices, he would pay x_0. Ex-ante, the expected value of the this payment in the auction with reserve price is exactly e.
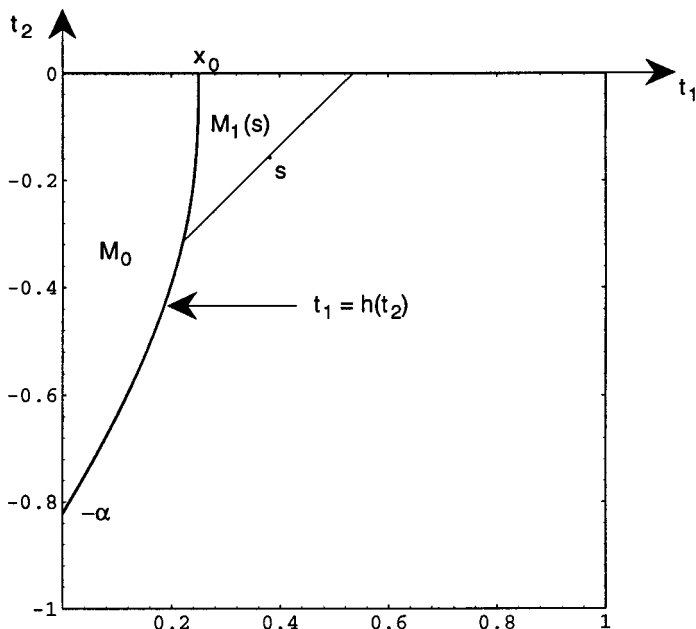
FIGURE 1

This is a system of first-order partial differential equations, whose solution yields the equilibrium for the second price auction with reserve price x_0.

We now illustrate the above technique for the case $N = 2$, where $T_1 = [0, v_1] \times [e_0, 0]$, $T_2 = [e_0, 0] \times [0, v_1]$, and $f_2(t_1, t_2) = f_1(t_2, t_1)$ for all $(t_1, t_2) \in T_2$. This is a negative externalities case, and thus we just need to consider nonnegative reserve prices x_0. Since by symmetry we only study buyer 1's strategy, the superindex for types will be omitted. Observe that the type $t = (x_0, 0)$, who suffers no externality, must be on the boundary of M_0 because he is indifferent between getting the object at price x_0 and not getting the object (and paying nothing). When x_0 is relatively small, the region M_0 is contained between the line segments $[0, x_0] \times \{0\}$, $\{0\} \times [-\alpha, 0]$, and the curve $t_1 = h(t_2)$, $t_2 \in [-\alpha, 0]$, between the points $(x_0, 0)$ and $(0, -\alpha)$, as shown in Fig. 1.[13] The observation above implies that $h(0) = x_0$. We will show that when x_0 is not too large, the solution of the differential equation satisfies $0 < h'(t_2) < 1$ for all $t_2 \in [-\alpha, 0]$. (Note that this implies that $\alpha > x_0$.)

[13] Figure 1 is drawn for $v_1 = 1$ and $e_0 = -1$. When x_0 is relatively large, $h(t_2) > 0$ for all $t_2 \in [e_0, 0]$, and if $e_0 > -\infty$, there exists $\alpha \in [0, v_1]$ such that the boundary of M_0 is made of the four segments: $[0, x_0] \times \{0\}$, $\{0\} \times [e_0, 0]$, $[0, \alpha] \times \{e_0\}$, and a curve from (α, e_0) to $(x_0, 0)$.

For each $t \in T_1$, let

$$M_1(t) := \{(s_1, s_2) \in T_1 \setminus M_0 \mid s_1 - s_2 \leqslant t_1 - t_2\}, \qquad \text{and}$$

$$m_1(t) := \text{Prob}(M_1(t)), \qquad \text{where} \quad \text{Prob}(A) = \int_A f_1(t^1) \, dt^1$$

denotes the measure of $A \subset \mathbf{R}^2$ according with the probability density f_1. Also define $m_0 := \text{Prob}(M_0)$, and $m(t_2) := m_1(h(t_2), t_2)$ for each $t_2 \in [-\alpha, 0]$.

If the players bid according to B, then each player's conditional probability assignment vector is piecewise continuous. Player 1's conditional probability assignment vector $q^1(t)$ is equal to

$$\begin{pmatrix} 0 \\ 1 - m_0 \end{pmatrix}$$

and

$$\begin{pmatrix} m_0 + m_1(t) \\ 1 - m_0 - m_1(t) \end{pmatrix}$$

for $t \in M_0$ and $t \in T_1 \setminus M_0$, respectively. Using the test proposed in footnote 4, for example, it is easy to check that $q^1(t)$ is conservative in each region M_0 and $T_1 \setminus M_0$. Moreover, $q^1(t)$ has a discontinuity at the boundary between regions M_0 and M_1. Consider a type $t = (t_1, t_2)$ in the boundary between M_0 and $T_1 \setminus M_0$. That is, such that $t_1 = h(t_2)$. Let $q^1(t^+)$ be the conditional probability assignment vector given that bidder 1 bids $t_1 - t_2$ (and bidder 2 uses strategy B_2). Similarly, let $q^1(t^-)$ be the conditional probability assignment vector if bidder 1 bids 0 instead. Then

$$\Delta q^1(t) := q^1(t^+) - q^1(t^-) = \begin{pmatrix} m_0 + m(t_2) \\ -m(t_2) \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ -h'(t_2) \end{pmatrix}$ is normal to the boundary of M_0 at t, and the condition that it is parallel to $\Delta q^1(t)$ (Proposition 2) becomes in this case

$$(m_0 = m(t_2)) \, h'(t_2) - m(t_2) = 0. \tag{1}$$

Since $m_0 > 0$ and $m(t_2) \geqslant 0$, Eq. (1) implies that $h'(t_2) \in [0, 1)$ for all $t_2 \in [-\alpha, 0]$, as was claimed earlier. The differential equation (1) assumes

that the parameter m_0 is known. However, it is related to the function h by

$$m_0 = \int_{-\alpha}^0 \int_0^{h(t_2)} f_1(t_1, t_2) \, dt_1 \, dt_2. \tag{2}$$

LEMMA 3. *Assume buyer* 2 *plays according to strategy* B_2. *Then, buyer* 1 *with type* $t \in T_1$ *is indifferent between bidding* $B_1^*(t)$ *and* 0 *iff* t *satisfies Eq.* (1) *and* $h(0) = x_0$.

The proof of Lemma 3 is in the Appendix.

PROPOSITION 6. *Assume that* (h, m_0) *solves the pair of Eqs.* (1) *and* (2), *with the initial condition* $h(0) = x_0$. *Then* B (*as defined above*) *is an equilibrium of the second price auction with reserve price* x_0.

Proof. Suppose $t \in T_1 \backslash M_0$, and let $b_1 = B_1^*(t)$. The argument that b_1 is a better response to B_2 than any $b_1' \neq b_1$ with $b_1' \in [x_0, 2]$ is exactly the same as in Proposition 4. On the other hand, Lemma 3 above establishes that Eq. (1) exactly characterizes the set of types $t \in T_1$ who are indifferent between bidding 0 and $B_1^*(t)$ given that buyer 2 uses the strategy B_2. Let $t \in T_1$ with $t_2 > -\alpha$ and $t_1 > h(t_2)$ (that is, $t_1 \in T_1 \backslash M_0$), and consider the type $t' := (h(t_2), t_2)$, which is on the boundary of M_0. Then, when both types t and t' bid $h(t_2) - t_2$, type t expects strictly more than type t'. Also, t' expects the same payoff when he bids $h(t_2) - t_2$ or 0. Finally, since $t_2' = t_2$, when types t and t' bid 0, they both expect the same payoff. These inequalities imply that type t strictly prefers bidding $B_1^*(t)$ to bidding 0. Similar arguments show that type $t \in T_1$ with $t_2 \leqslant -\alpha$ strictly prefers bidding $B_1^*(t)$ to bidding 0 (in this case, consider the type $t' = (0, -\alpha)$ for the argument), and that a type $t \in M_0$ strictly prefers the bid 0 to any bid above x_0. ∎

EXAMPLE. Let $v_1 = 1$, $e_0 = -1$, and $f_1 \equiv 1$ in $[0, 1] \times [-1, 0]$ (that is, f_1 is the density of a uniform distribution). In this case, there is a unique solution of Eqs. (1) and (2), which is given by

$$h(t_2) = t_2 - \sqrt{m_0} \ln(t_2 + \sqrt{m_0 + t_2^2}) + x_0 + \sqrt{m_0} \ln(\sqrt{m_0}),$$

where

$$c_1 = \sqrt{2 + 2\sqrt{2}}, \qquad c_2 := [c_1 + \ln(-c_1 + \sqrt{1 + c_1^2})]^{-1}$$
$$m_0 = (c_2 x_0)^2 = 2.23569 x_0^2, \qquad \text{and} \qquad \alpha = c_1 c_2 x_0 = 3.28555 x_0.$$

The intermediate steps are provided in the Appendix.

Optimality and exclusion. We now wish to use the example above to provide some insights for the optimality of reserve prices in our multidimensional setting with externalities. In a symmetric model without externalities, Myerson has shown that the optimal auction is a second-price auction with a reserve price. The seller may announce a reserve price that is higher than her true reservation value, in which case the seller excludes some types of buyers from consumption. Exclusion in Myerson's one-dimensional setting has two effects that go in opposite directions. By excluding some types: (1) the seller restricts supply, and is able to collect higher prices; (2) the seller foregoes potentially profitable trades. In standard one-dimensional problems, the relative strength of those effects is determined by the relation between the seller's and buyers' valuations.

For his model of a multiproduct monopolist, Armstrong proves that the first effect is always dominant. The intuition is as follows. Consider the constrained optimal pricing policy in which all consumers must be served. Naturally, with this policy there are consumers with "low" valuations who are just indifferent between buying a bundle and nonparticipation. The seller can then raise its total revenues by increasing uniformly the price of every bundle it offers by a small amount $\varepsilon > 0$. With the new pricing policy, all the consumers who still participate, buy the same bundle as before, but pay ε more. The new pricing policy excludes those consumers who before were obtaining a surplus less than ε. However they represent a fraction of the total mass of consumers of order $o(\varepsilon)$.[14] Thus, exclusion of a fraction of consumers is always optimal, even if the consumers' valuations are much higher than the production costs.

In our model, however, there is a third effect due to competition: when the seller excludes some types, buyers are less afraid that the good might go to another buyer. Consequently, nonparticipation becomes more attractive, and the seller must decrease the revenue it extracts from each type of every participating buyer.

LEMMA 4. *There exists $\delta > 0$ such that the optimal reserve price is $x_0 = 0$ for all t^0 with $t_0^0 - t_1^0 = t_0^0 - t_2^0 \leqslant \delta$. Moreover, a reserve price in the interval $(0, 0.2071]$ is never optimal. That is, if it is optimal to set a positive reserve price for some value of $t_0^0 - t_1^0$, it must be that $x_0 > 0.2071$.*

The proof of Lemma 4 is in the Appendix; here we provide some intuition. Consider the effects of increasing the reserve price from 0 to $x_0 = \varepsilon$, for a small $\varepsilon > 0$: (a) the seller now keeps the object when both bidders' types are in M_0; (b) the seller now gets ε instead of $\min \{B_1^*(t^1), B_2^*(t^2)\}$ when one bidder's type is in M_0 and the other bidder's type is outside M_0. As in

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[14] A function $h(\varepsilon)$ is order $o(\varepsilon)$ if $\lim_{e \to 0} h(\varepsilon)/\varepsilon = 0$.

Armstrong's paper, the first effect is of smaller order than the second (the orders are ε^4 and ε^3 respectively—see the proof of Lemma 4). Hence, the valuation of the seller plays no role in determining whether a small amount of exclusion is desirable. However, in contrast to the setting where a multi-product monopolist faces a measure of non-interacting buyers, the sign of the second effect is ambiguous. This is due to the additional competition effect among buyers. The sign of this term depends on whether the average bid $B_i^*(t^i)$ of types in M_0 (who are excluded when the reserve price ε is imposed) is below or above ε.[15] In our example, it turns out that this average bid is *above* ε, and hence exclusion is suboptimal. Compared to Myerson's result for the standard case, this result by itself is not surprising: if the seller cost is sufficiently lower than the minimum buyer's valuation, in his model too the optimal reserve price is 0. However, contrary to his case, here the optimal reserve price is not a continuous function of the difference $t_0^0 - t_1^0 = t_0^0 - t_2^0$. As this difference increases, the optimal reserve price function eventually jumps from 0 to a strictly positive value.

Consider now the class of standard bidding mechanisms[16] satisfying the following property. For each player i, there exists a set $X_i \subset T_i$ such that $\sum_i p_i(t^1, ..., t^N) = 0$ if $t^i \in X_i$ for all i, and $\sum_i p_i(t^1, ..., t^N) = 1$ otherwise. Observe that this class of mechanisms covers the type of exclusion achieved by instruments such as reserve prices and entry fee.[17] For this class, it can be shown that the second-price auction with a (possibly zero) reserve price is the revenue maximizing mechanism. Thus, in this class, exclusion is not always optimal. The argument relies on the following observations: (1) in the region where the object is transferred with probability 1, the object must be allocated to the player with the highest summary type $B_i^*(t^i)$; (2) the monotonicity and integrability conditions imply that for any player i, the set X_i coincides with the set M_0 for some reserve price x_0.

7. APPENDIX

Proof of Proposition 2. Since the set of piecewise constant functions is dense in the set of piecewise continuous functions (with the sup norm), it is enough to prove the result for piecewise constant functions. Suppose then that q^i is piecewise constant: $\{M_1, ..., M_k\}$ is a partition of T_i, and for each j, there exists $z^j \in \mathbf{R}^N$ such that $q^i(t^i) = z^j$ for all $t^i \in M_j$. Let M_j and M_m be two adjacent regions, and for each $s^i \in M_j \cap M_m$, let $n(s^i)$ denote the

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[15] The answer to this is not obvious because of the shape of the exclusion area M_0.

[16] For the case $N = 2$, the remark at the end of Section 5 can be used to remove the restriction to standard mechanisms.

[17] This is also the type of exclusion considered in models of multiproduct monopoly pricing where agents do not interact.

unitary exterior normal to the boundary of M_j at s^i. Thus, for sufficiently small $\varepsilon > 0$, $s^i + \varepsilon n(s^i)$ is in the interior of M_m and $s^i - \varepsilon n(s^i)$ is in the interior of M_j. Pick any two points $s^i, t^i \in M_j \cap M_m$, and let Γ be a piecewise continuously differentiable path from s^i to t^i completely contained in $M_j \cap M_m$. That is, $\Gamma: [0, 1] \to M_j \cap M_m$ is a piecewise continuously differentiable function such that $\Gamma(0) = s^i$ and $\Gamma(1) = t^i$. For sufficiently small $\varepsilon > 0$, define the two paths $\Gamma_j: [0, 1] \to M_j$ and $\Gamma_m: [0, 1] \to M_m$ by

$$\Gamma_j(x) := \Gamma(x) - \varepsilon n(\Gamma(x)) \qquad \text{and} \qquad \Gamma_m(x) := \Gamma(x) + \varepsilon n(\Gamma(x)),$$

for each $x \in [0, 1]$. Also, let Γ_{jm} be the straight path from $s^i - \varepsilon n(s^i)$ to $s^i + \varepsilon n(s^i)$, and Γ_{mj} be the straight path from $t^i + \varepsilon n(t^i)$ to $t^i - \varepsilon n(t^i)$. Then the path formed by concatenating the paths Γ_{jm}, Γ_m, Γ_{mj} and $-\Gamma_j$, in that order, is a closed path. Since q^i is conservative, its line integral along this path must be 0. As $\varepsilon \to 0$, this implies that

$$\int_0^1 \Delta q^i(\Gamma(x)) \cdot \Gamma'(x) \, dx = 0.$$

Since this is true for every path Γ in the common boundary between M_j and M_m, we must have that for every $t^i \in M_j \cap M_m$, $\Delta q^i(t^i)$ is perpendicular to any vector tangent to the common boundary $M_j \cap M_m$ at t^i. That is, $\Delta q^i(t^i)$ must be a multiple of $n(t^i)$. ∎

Proof of Proposition 3. As we explained before the statement of Proposition 3, when $y(C) = y^*(C)$, $S_i(O^i) = \beta := \min\{e_1, 0\}$. Since q^i is monotone, S_i is convex, and $S_i(t^i) \geqslant S_i(O^i) + q^i(O^i) \cdot (t^i - O^i)$ for all $t^i \in T_i$ (because $q^i(O^i) \in \partial S_i(O^i)$). Therefore

$$S_i(t^i) \geqslant \beta + q^i(O^i) \cdot (t^i - O^i) \qquad \text{for all} \quad t^i \in T_i.$$

Thus, it suffices to show that

$$\beta + q^i(O^i) \cdot (t^i - O^i) \geqslant \rho^i \cdot t^i \qquad \text{for all} \quad t^i \in T_i. \tag{3}$$

For the proof of case (i), when $e_1 < 0$, choose for example

$$\rho^i := \begin{cases} \dfrac{1}{\sum_{j \neq i} q_j^i(C^i)} (q_1^i(C^i), ..., q_{i-1}^i(C^i), 0, q_{i+1}^i(C^i), ..., q_N^i(C^i)) \\ \qquad\qquad\qquad \text{if} \quad q_i^i(C^i) < 1 \\ \dfrac{1}{N}(1, ..., 1, 0, 1, ..., 1) \qquad \text{if} \quad q_i^i(C^i) = 1. \end{cases}$$

Recall that in this case $\beta = e_1$ and $O^i = C^i$. Since $\rho^i \cdot C^i = e_1$, inequality (3) is equivalent to $(q^i(C^i) - \rho^i) \cdot (t^i - C^i) \geq 0$. But $q_i^i(C^i) \geq 0 = \rho_i^i$ and $t_i^i \geq v_0 = C_i^i$, and for $j \neq i$, $q_j^i(C^i) \leq \rho_j^i$ and $t_j^i \leq e_1 = C_j^i$. Therefore

$$(q^i(C^i) - \rho^i) \cdot (t^i - C^i) = \sum_{j \in I} (q_j^i(C^i) - \rho_j^i)(t_j^i - C_j^i) \geq 0.$$

In case (ii), when $e_0 > 0$, $\beta = 0$, and $\rho^i \cdot t^i = 0$ for each $t^i \in T_i$. Also, $t_i^i \geq v_0 = O_i^i$, and $t_j^i \geq e_0 = O_j^i$ for all other j. And since $q_j^i(O^i) \geq 0$ for all j, $q^i(O^i) \cdot (t^i - O^i) \geq 0$, as desired.

Finally, in case (iii), when $e_0 \leq 0 \leq e_1$, $\beta = 0$, and $\rho^i \cdot O^i = 0$. Therefore, inequality (3) becomes again $(q^i(O^i) - \rho^i) \cdot (t^i - O^i) \geq 0$, which by the definition of ρ^i is equivalent to

$$q_i^i(O^i)(t_i^i - v_0) \geq 0.$$

The last inequality is satisfied because $t_i^i \geq v_0$. ∎

Proof of Lemma 1. To prove the first equality, it is enough to show that for any $b_1 \in [\underline{b}, \bar{b}]$ and $1 < i < j$,

$$q_i^1(b_1) = q_j^1(b_1).$$

To simplify the notation, we prove this equality for the case $i = 2$ and $j = 3$. Let $b_1 \in [\underline{b}, \bar{b}]$; then, by (A_2),

$$q_2^1(b_1) = \int_{T_{-1}} p_2(b_1, B_1(\Pi_{12}(t^2)), ..., B_1(\Pi_{1N}(t^N))) \, f_{-1}(t^{-1}) \, dt^{-1}$$

$$= \int_{T_{-1}} p_3(b_1, B_1(\Pi_{13}(t^3), B_1(\Pi_{12}(t^2), ..., B_1(\Pi_{1N}(t^N)))$$

$$\times f_{-1}(t^{-1}) \, dt^{-1} =: \alpha,$$

Since

$$p_3(b_1, B_1(\Pi_{13}(t^3)), B_1(\Pi_{12}(t^2)), ..., B_1(\Pi_{1N}(t^N))) \, f_3(t^3) \, f_2(t^2)$$

$$= p_3(b_1, B_1(\Pi_{13}(t^3)), B_1(\Pi_{12}(t^2)), ..., B_1(\Pi_{1N}(t^N)))$$

$$\times f_2(t_1^3, t_3^3, t_2^3, ...) \, f_3(t_1^2, t_3^2, t_2^2, ...)$$

$$= p_3(b_1, B_1(\Pi_{13}(t^3)), B_1(\Pi_{12}(t^2)), ..., B_1(\Pi_{1N}(t^N)))$$

$$\times f_2(t_2^3, t_3^3, t_1^3, ...) \, f_3(t_3^2, t_1^2, t_2^2, ...)$$

$$= p_3(b_1, B_1(\Pi_{12}(s^2)), B_1(\Pi_{13}(s^3)), ..., B_1(\Pi_{1N}(s^N))) \, f_2(s^2) \, f_3(s^3),$$

where $s^2 := (t_2^3, t_3^3, t_1^3, ...)$, $s^3 := (t_3^2, t_1^2, t_2^2, ...)$, and $s^i := t^i$ for all $i > 3$, we have that

$$\alpha = \int_{T_{-1}} p_3(b_1, B_1(\Pi_{12}(s^2)), B_1(\Pi_{13}(s^3)), ..., B_1(\Pi_{1N}(s^N)))$$

$$\times f_{-1}(s^{-1}) \, ds^{-1} = q_3^1(b_1).$$

We have then shown that there is some $\alpha \in [0, 1]$ such that $q_i^1(b_1) = \alpha$ for all $i > 1$. Since the coordinates of $q^1(b_1)$ add up to 1, we must have that $q_0^1(b_1) + q_1^1(b_1) + (N-1) \alpha = 1$; this shows the first equality. The second equality is shown analogously. ∎

Proof of Lemma 3. For any $t_2 \in [-\alpha, 0]$, consider the type $(h(t_2), t_2)$ in the boundary between M_0 and $T_1 \backslash M_0$. Let $\Phi(t_2)$ be the difference in expected payoff for that type when he bids $t_1 - t_2$ and when he bids 0. That is,

$$\Phi(t_2) := (1 - m_0 - m(t_2)) \, t_2 + (m_0 + m(t_2)) \, t_1$$

$$- \left(m_0 x_0 + \int_{M_1(h(t_2), t_2)} (s_1 - s_2) \, f_1(s) \, ds \right) - (1 - m_0) \, t_2$$

$$= m_0(t_1 - x_0) + m(t_2)(t_1 - t_2) - \int_{M_1(h(t_2), t_2)} (s_1 - s_2) \, f_1(s) \, ds$$

$$= m_0(h(t_2) - x_0) + m(t_2)(h(t_2) - t_2)$$

$$- \int_{t_2}^0 \int_{h(s_2)}^{h(t_2) - t_2 + s_2} (s_1 - s_2) \, f_1(s) \, ds.$$

To prove the lemma, we need to show that $\Phi(t_2) = 0$ for all $t_2 \in [-\alpha, 0]$. Since $h(0) = x_0$ and $m(0) = 0$, we have that $\Phi(0) = 0$. Hence, $\Phi(t_2) = 0$ for all $t_2 \in [-\alpha, 0]$ is equivalent to $\Phi'(t_2) = 0$ for all $t_2 \in [-\alpha, 0]$. Define $\Psi(s_2, t_2) := \int_{h(s_2)}^{h(t_2) - t_2 + s_2} (s_1 - s_2) \, f_1(s) \, ds_1$. Then

$$\frac{d}{dt_2} \int_{t_2}^0 \Psi(s_2, t_2) \, ds_2 = -\Psi(t_2, t_2) + \int_{t_2}^0 \frac{\partial \Psi}{\partial t_2}(s_2, t_2) \, ds_2.$$

Since $\Psi(t_2, t_2) = 0$ and $(\partial \Psi / \partial t_2)(s_2, t_2) = (h(t_2) - t_2) \, f_1(h(t_2) - t_2 + s_2, s_2)$ $\times (h'(t_2) - 1)$, we have

$$\Phi'(t_2) = m_0 h'(t_2) + m'(t_2)(h(t_2) - t_2) + m(t_2)(h'(t_2) - 1)$$
$$- \frac{d}{dt_2} \int_{t_2}^0 \Psi(s_2, t_2) \, ds_2$$
$$= m_0 h'(t_2) + m'(t_2)(h(t_2) - t_2) + m(t_2)(h'(t_2) - 1)$$
$$- (h(t_2) - t_2)(h'(t_2) - 1) \int_{t_2}^0 f_1(h(t_2) - t_2 + s_2, s_2) \, ds_2$$
$$= [(m_0 + m(t_2)) \, h'(t_2) - m(t_2)] + (h(t_2) - t_2)$$
$$\times \left[m'(t_2) - (h'(t_2) - 1) \int_{t_2}^0 f_1(h(t_2) - t_2 + s_2, s_2) \, ds_2 \right].$$

One can check using the same differentiation technique as above that

$$m'(t_2) = \frac{d}{dt_2} \int_{t_2}^0 \int_{h(s_2)}^{h(t_2) - t_2 + s_2} f_1(s_1, s_2) \, ds_1 \, ds_2$$
$$= (h'(t_2) - 1) \int_{t_2}^0 f_1(h(t_2) - t_2 + s_2, s_2) \, ds_2.$$

Therefore,

$$\Phi'(t_2) = (m_0 + m(t_2)) \, h'(t_2) - m(t_2),$$

which is 0 iff h solves the differential equation (1). Thus, the condition that each type $t = (h(t_2), t_2)$, $t_2 \in [-\alpha, 0]$, be indifferent between bidding $t_1 - t_2$ and bidding 0 is equivalent to the differential equation (1). ∎

Solution of differential equation (1). We now solve the differential equation (1) for the case in which $v_1 = 1 = -e_0$ and f_1 represents the uniform distribution in $[0, 1] \times [-1, 0]$. By definition

$$m(t_2) = \int_{t_2}^0 \int_{h(s_2)}^{h(t_2) - t_2 + s_2} f_1(s_1, s_2) \, ds_1 \, ds_2$$
$$= \int_{t_2}^0 [h(t_2) - t_2 + s_2 - h(s_2)] \, ds_2 = \int_{t_2}^0 [k(t_2) - k(s_2)] \, ds_2,$$

where $k(t_2) := h(t_2) - t_2$. Note that $k'(t_2) = h'(t_2) - 1$ and that $m'(t_2) = -t_2 k'(t_2)$. Therefore, the differential equation (1) becomes

$$(m_0 + m(t_2))(t_2 - m'(t_2)) - t_2 m(t_2) = 0. \tag{4}$$

The solution of (4) is given by

$$m(t_2) = -m_0 + \sqrt{m_0^2 + m_0 t_2^2 + c},$$

for some constant c. Since $m(0) = 0$, $c = 0$. Now,

$$k'(t_2) = \frac{-m'(t_2)}{t_2} = \frac{-m_0}{\sqrt{m_0^2 + m_0 t_2^2}},$$

and therefore

$$k(t_2) = \sqrt{m_0} \left[\ln(\sqrt{m_0}) - \ln(t_2 + \sqrt{m_0 + t_2^2}) \right] + c,$$

where c is another integration constant. Finally, since $h(t_2) = t_2 + k(t_2)$ and $h(0) = x_0$, we have that $c = x_0$ and

$$h(t_2) = t_2 - \sqrt{m_0} \ln(t_2 + \sqrt{m_0 + t_2^2}) + x_0 + \sqrt{m_0} \ln(\sqrt{m_0}).$$

To determine the parameters m_0 and α, we use the conditions: $m_0 = \int_{-\alpha}^{0} h(t_2)\, dt_2$ and $h(-\alpha) = 0$. That is

$$m_0 = (x_0 + \sqrt{m_0} \ln(\sqrt{m_0})) \alpha - \frac{\alpha^2}{2} - \sqrt{m_0} \int_{-\alpha}^{0} \ln(t_2 + \sqrt{m_0 + t_2^2})\, dt_2 \qquad (5)$$

$$0 = x_0 + \sqrt{m_0} \ln(\sqrt{m_0}) - \alpha - \sqrt{m_0} \ln(-\alpha + \sqrt{m_0 + \alpha^2}). \qquad (6)$$

Since

$$\int \ln(t_2 + \sqrt{m_0 + t_2^2})\, dt_2 = -\sqrt{m_0 + t_2^2} + t_2 \ln(t_2 + \sqrt{m_0 + t_2^2}),$$

Eq. (5) becomes

$$0 = (x_0 + \sqrt{m_0} \ln(\sqrt{m_0})) \alpha - \frac{\alpha^2}{2}$$
$$- \sqrt{m_0}[\sqrt{m_0 + \alpha^2} + \alpha \ln(-\alpha + \sqrt{m_0 + \alpha^2})]. \qquad (5')$$

Multiplying (6) by α and subtracting the result from (5') yields

$$0 = \frac{\alpha^2}{2} - \sqrt{m_0} \sqrt{m_0 + \alpha^2} \qquad \text{or equivalently}$$

$$0 = \alpha^4 - 4m_0 \alpha^2 - 4m_0^2. \qquad (7)$$

The relevant solution of (7) is

$$\alpha = c_1 \sqrt{m_0} \qquad \text{where} \quad c_1 := \sqrt{2 + 2\sqrt{2}}.$$

Finally, let $c_2 := [c_1 + \ln(-c_1 + \sqrt{1 + c_1^2})]^{-1} = 1.49522$ and substitute α in (6) to get

$$m_0 = (c_2 x_0)^2 = 2.23569 x_0^2, \qquad \text{and}$$

$$\alpha = c_1 c_2 x_0 = 3.28555 x_0,$$

which are valid provided $0 \leqslant x_0 \leqslant 0.30436 = (c_1 c_2)^{-1}$ (since our analysis assumed that $\alpha \leqslant 1$).

Proof of Lemma 4. For each reserve price x_0, we now compute the seller's expected revenue. Let the random variables X and B denote respectively the seller's revenue and player 1's bid. For each $x \in [0, 2]$, let $F_X(x) := \text{Prob}[X \leqslant x]$ and $F_B(x) := \text{Prob}[B \leqslant x]$. Finally, let $\ell := k^{-1}$ (where $k(t_2) = h(t_2) - t_2$ and h is the function that parametrizes the boundary of M_0). For any $x \in [x_0, \alpha]$, the set of types for player 1 that bid less than or equal to x is $M_0 \cup M(t_2)$, where t_2 is such that $x = h(t_2) - t_2 = k(t_2)$. Therefore

$$F_B(x) = \begin{cases} m_0 & \text{if} \quad 0 \leqslant x < x_0 \\ m_0 + m(\ell(x)) & \text{if} \quad x_0 \leqslant x \leqslant \alpha \\ p^2/2 & \text{if} \quad \alpha \leqslant x \leqslant 1 \\ 1 - (2-p)^2/2 & \text{if} \quad 1 \leqslant x \leqslant 2. \end{cases}$$

Let f_X and f_B denote respectively the Radon–Nikodym derivative of the absolutely continuous parts of F_X and F_B. Since

$$F_X(x) = [F_B(x)]^2 + 2F_B(x)[1 - F_B(x)] = F_B(x)[2 - F_B(x)],$$

we have

$$\begin{aligned} f_X(x) &= 2[1 - F_B(x)] f_B(x) \\ &= \begin{cases} 0 & \text{if} \quad 0 \leqslant x < x_0 \\ 2(1 - m_0 - m(\ell(x))) \, m'(\ell(x)) \, \ell'(x) & \text{if} \quad x_0 \leqslant x \leqslant \alpha \\ (2 - p^2) \, p & \text{if} \quad \alpha \leqslant x \leqslant 1 \\ (2 - p)3 & \text{if} \quad 1 \leqslant x \leqslant 2. \end{cases} \end{aligned}$$

Note that by Eq. (2), $(m_0 + m(t_2)) \, m'(t_2) = m_0 t$ for all $t_2 \in [-\alpha, 0]$. Hence, the seller's expected revenue is

$$
\begin{aligned}
E[X] &= 2m_0(1-m_0) \, x_0 + \int_{x_0}^{2} p f_X(x) \, dx \\[2mm]
&= 2m_0(1-m_0) \, x_0 + \int_{x_0}^{\alpha} 2p(1-m_0-m(\ell(x))) \, m'(\ell(x)) \, \ell'(x) \, dx \\[2mm]
&\quad + \int_{\alpha}^{1} x^2(2-x^2) \, dx + \int_{1}^{2} x(2-x)^3 \, dx \\[2mm]
&= 2m_0(1-m_0) \, x_0 + \int_{x_0}^{\alpha} 2x[m'(\ell(x)) - m_0\ell(x)] \, \ell'(x) \, dx \\[2mm]
&\quad + \frac{7-10\alpha^3+3\alpha^5}{15} + \frac{3}{10} \\[2mm]
&= 2m_0(1-m_0) \, x_0 + \int_{0}^{-\alpha} 2k(t)[m'(t)-m_0 t] \, dt + \frac{23}{30} + \frac{3\alpha^5-10\alpha^3}{15} \\[2mm]
&= 2m_0(1-m_0) \, x_0 - 2m_0 \int_{-\alpha}^{0} tk(t) \left[\frac{1}{\sqrt{m_0^2+m_0 t^2}} - 1 \right] dt \\[2mm]
&\quad + \frac{23}{30} + \frac{3\alpha^5-10\alpha^3}{15}.
\end{aligned}
$$

In the previous to the last line we have used the change of variables $t = \ell(x)$, and the fact that $h(-\alpha) = 0$ implies that $k(-\alpha) = \alpha$ and $h(0) = x_0$ implies $k(0) = x_0$.

Albeit complex, the last integral can be computed explicitly in closed form, and we can express the expected revenue as a function of the reserve price as follows:

$$
E[X] = \tfrac{23}{30} - 2.86852 x_0^3 + 1.39374 x_0^5.
$$

$E[X]$ is a decreasing and concave function of x_0 in the interval $[0, 0.30436]$, and its slope is 0 at $x_0 = 0$. This can be seen directly in Fig. 2, where we plot $E[X]$ as a function of x_0.

Although our explicit analysis applies for $x_0 \in [0, 0.30436]$ only, one can show that $E[X]$ is decreasing in x_0 throughout the whole relevant interval $[0, 2]$.
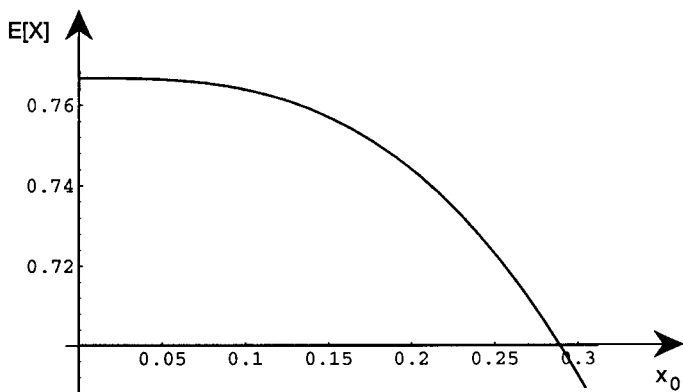
FIGURE 2

Since the seller keeps the object when both buyers' types are in M_0, and this happens with probability m_0^2, the seller's total expected payoff when $x_0 \in [0, 0.30436]$ is

$$E[X] + t_1^0 + m_0^2(t_0^0 - t_1^0)$$

$$= t_1^0 + \frac{23}{30} - 2.86852x_0^3 + 4.99831(t_0^0 - t_1^0) x_0^4 + 1.39374x_0^5.$$

The externality term $m_0^2(t_0^0 - t_1^0) = 4.99831(t_0^0 - t_1^0) x_0^4$ is convex and increasing in x_0. But, it is also flat at $x_0 = 0$. Since this term changes at a slower rate (x_0^4) than the expected revenue $(-x_0^3)$ near $x_0 = 0$, the latter term dominates the changes in the seller's total expected payoff for x_0 near 0, for any value of $t_0^0 - t_1^0$. Moreover, for $t_0^0 - t_1^0 = 2$, the optimal reserve price is clearly $x_0 = 2$. Now, for any $t_0^0 - t_1^0 \leqslant 2$, the externality term $m_0^2(t_0^0 - t_1^0)$ is bounded above by $10x_0^4$ (and its derivative is bounded above by $40x_0^3$). Thus,

$$\frac{d}{dx_0} E[X] \leqslant x_0^2(7x_0^2 + 40x_0 - 8.6) < 0$$

for all $x_0 \in [0, 0.2071]$. Hence, $x_0 \in (0, 0.2071]$ is never optimal, independent of the value of $t_0^0 - t_1^0$. ∎

REFERENCES

1. M. Armstrong, Multiproduct nonlinear pricing, *Econometrica* **64** (1996), 51–75.
2. P. Champsaur and J. C. Rochet, Multiproduct duopolists, *Econometrica* **57** (1989), 533–588.

3. P. Jehiel, B. Moldovanu, and E. Stacchetti, How (not) to sell nuclear weapons, *Amer. Econ. Rev.* **86** (1996), 814–829.

4. V. Krishna and E. Maenner, A path independence result for convex functions with an application to mechanism design, mimeo, Penn State University, (1998).

5. J. J. Laffont, E. Maskin, and J. C. Rochet, Optimal nonlinear pricing with two-dimensional characteristics, *in* "Information, Incentives, and Economic Mechanisms" (Groves *et al.*, Eds.), pp. 256–266, University of Minnesota Press, Minneapolis, 1987.

6. J. J. Laffont and J. Tirole, The regulation of multiproduct firms, Part I: Theory, *J. Public Econ.* **43** (1990), 1–36.

7. T. Lewis and D. Sappington, Regulating a monopolist with unknown demand and cost functions, *Rand J. Econ.* **19** (1988), 438–457.

8. S. Matthews and J. Moore, Monopoly provision of quality and warranties: An exploration in the theory of multidimensional screening, *Econometrica* **55** (1987), 441–468.

9. R. P. McAfee and J. McMillan, Multidimensional incentive compatibility and mechanism design, *J. Econ. Theory* **46** (1988), 335–354.

10. R. P. McAfee, J. McMillan, and M. D. Whinson, Multiproduct monopoly, commodity bundling, and correlation of values, *Quart. J. Econ.* **93** (1989), 371–383.

11. P. Milgrom and R. Weber, A theory of auctions and competitive bidding, *Econometrica* **50** (1982), 1089–1122.

12. L. J. Mirman and D. Sibley, Optimal nonlinear prices for multiproduct monopolies, *Bell. J. Econ.* **11** (1980), 659–670.

13. R. Myerson, Optimal auction design, *Math. Oper. Res.* **6** (1981), 58–63.

14. T. R. Palfrey, Bundling decisions by a multiproduct monopolist with incomplete information, *Econometrica* **51** (1983), 463–483.

15. J. C. Rochet, The taxation principle and multitime Hamilton–Jacobi equations, *J. Math. Econ.* **14** (1985), 93–128.

16. J. C. Rochet and P. Choné, Ironing, sweeping and multidimensional screening, *Econometrica* **66** (1998), 783–826.

17. R. T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, 1972.

18. R. T. Rockafellar, "Conjugate Duality and Optimization," SIAM, Philadelphia, 1974.

19. A. M. Spence, Multi-product quantity-dependent prices and profitability constraints, *Rev. Econ. Stud.* **47** (1980), 821–841.

20. R. Wilson, Multiproduct tariffs, *J. Regulatory Econ.* **3** (1991), 5–26.

21. R. Wilson, "Nonlinear Pricing," Oxford Univ. Press, Oxford, 1993.