Optimal Security Design for Risk-Averse Investors

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Abstract

We use the tools of mechanism design, combined with the theory of risk measures, to analyze how a cash constrained owner of an asset with known, stochastic returns raises capital from a population of investors that differ in their risk aversion and budget constraints. The issuer partitions the asset's cash flow into several asset-backed securities, one for each type of investor. The optimal partition conforms to the commonly observed practice of tranching into senior debt, junior debt and equity. Tranching endogenously arises due to the differences in risk appetites among agents, and in the budget constraints they face.

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1 Introduction

Even after the financial crisis, securitization remains a multi-trillion-dollar business. It is employed by entrepreneurs to fund projects with uncertain returns or by banks to enhance capital capacity. The underlying asset in securitization is typically a pool of financial obligations, such as mortgages or loans, but it can also be a cash-flow-generating fixed asset, such as a ship, an aircraft, or an entire business. The profile of expected cash flows from the underlying asset is synthetically partitioned and sold into multiple *tranches*.[1] These tranches, all backed by the same pool of assets, exhibit different risk, yield, duration, and other characteristics. The defining feature of observed tranching in practice is that each additional dollar of cash flow is allocated to a unique type of tranche in decreasing order of subordination: payments to investors conform to a *waterfall* structure where more senior claims must be fully paid before junior ones start to be served. As Gorton and Metrick [2013] note in their handbook article on securitization: "The tranching of pools sold to Special Purpose Vehicles remains a puzzle."

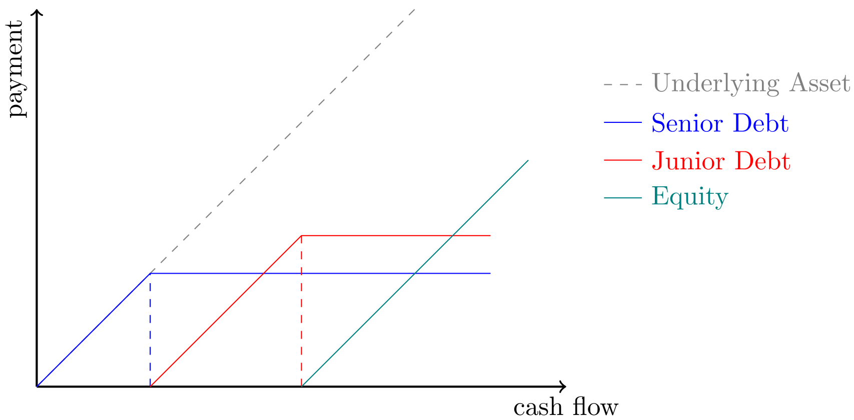

Figure 1: Illustration of Tranching

In the present paper, we use the tools of mechanism design, combined with the theory of risk measures, to analyze a model where an issuer with insufficient financial resources raises capital from a population of different classes of risk-averse and budget-constrained investors by securitizing an underlying asset that has a stochastic return. The investors are heterogeneous in their risk preferences and privately informed about both their preferences and budgets. The issuer then employs a menu of securities to raise funds.

In order to clearly differentiate our explanation from the classical one based on

[1] For example, Cummings et al. [2020] document a huge data set containing more than 150,000 syndicated loans across the world. About 30% of those were tranched, with an average of 2.4 tranches per issue.

a lemons problem where the issuer has more information than the investor about the asset's returns, the distribution of the project's cash flow is assumed here to be common-knowledge: no agent has private information about it.

Another main methodological departure from the classical finance literature where investors are expected utility maximizers arises from our assumption that investors assess risk using the large and important class of *spectral* or *coherent distortion* risk measures (see, for example, Acerbi [2002] and Wang [1996]).[2] This class is obtained by taking weighted sums of *average values at risk* (or *expected shortfalls),* and it forms the building block for the entire set of law-invariant, *coherent* risk measures described and axiomatized in a famous paper by Artzner et al. [1999].[3] Thus, the risk measurement we use here closely follows the most recent recommendations made by the *Basel Committee on Banking Supervision* [2019], the primary global standard setter for the prudential regulation of financial institutions, or by the widely used *Swiss Solvency Test* for insurance companies [2024]. These guidelines point out the main deficiency of the ubiquitous Value of Risk (VaR) measure – specifically, that it may penalize portfolio diversification – and establish the expected shortfall (ES) as the preferred way of assessing risks in financial markets.

Our main results show that, in the optimal mechanism, the issuer partitions and sells the underlying asset's realized cash flow into several tranches of debt or equity, one for each type of risk-averse investor, such that more conservative investors are offered less risky securities. The optimal tranching exactly conforms to the commonly observed "waterfall" structure that features a subordinated payment scheme: senior debt, possibly junior debt, and equity.

In a frictionless, complete market nature of issued securities should be irrelevant: the field of security design aims to explain optimal financial structures given prevalent market frictions. The standard theoretical argument is that tranching occurs because it enables a decomposition of the asset's cash flow into a component that is largely independent of a seller's private information (debt), and an information sensitive component with cash flows that are dependent on the seller's information (equity). Although this theory predicts which security—debt or equity—is retained by the issuer, it does not explain how or why the various sold securities are allocated to heterogeneous investors. Meanwhile, the rise and growing significance of publicly available credit ratings for securities, combined with regulated information disclosure

[2]A coherent risk measure is a function from random variables to the reals that satisfies monotonicity, sub-additivity, homogeneity, and translational invariance. See also Rüschendorf [2013] for the meaning of these properties when applied to risk measures.

[3]The class of law invariant, coherent risk measures is obtained by minimizing over several possible distortions. See Safra and Segal [1998] and Kusuoka [2001]. In the special case where the project to be financed can either succeed or fail, our results are more general and can be applied to the entire class of such risk measures.

in sales prospectuses, help mitigate the aforementioned lemons problem.

Our model explains the emergence of tranching without appealing to an asymmetry of information about the quality of the underlying asset. Tranching endogenously arises here in an optimal mechanism because of simpler and more basic economic forces, namely the differences in risk appetites among investors. Indeed, a principal practical motivation for securitization is to appeal to investor groups with heterogeneous preferences: tranching caters to both conservative and more aggressive (i.e., less risk-averse) investors since it provides a variety of product choices tailored to specific investor needs in terms of duration, risks, cash-flow patterns, and yields.[4] Senior securities are designed to appeal to "constrained" or *conservative* investors who can only purchase investment-grade products, and are thus more risk-averse. These investors are bound by either transparent regulations (e.g. banks, pension funds, and insurance companies are often restricted in the types of assets they may hold), or by less transparent constraints such as internal by-law restrictions/investment mandates or other portfolio/time-specific hedging requirements.[5] Less constrained, *aggressive* investors - such as hedge funds, private equity funds and sovereign funds - are less risk averse, and are thus willing to purchase riskier securities. The junior securities that offer higher returns in exchange for more risk are designed to appeal to them.

Since the reasons for being constrained in the above sense are diverse and also change over time, the same financial institution may belong to either group in a specific transaction, and the issuer may not know what is the current risk appetite of an individual investor, nor what his present investment budget is. For example, although the current Basel Framework recommends a standard approach (based, among other factors, on expected shortfall as a measure of risk), it still allows financial institutions to use an internal, private risk models for up to 40% of their portfolio. Even if the issuer possesses information about the various risk preferences, regulatory constraints or the possibility of resale may prevent her from effectively using it. We illustrate the last point further in Section 6.1, where we consider the possibility of trading among investors after the initial sale of securities by the issuer.[6]

[4]For example, Fuster et al [2022] classify the investors in US mortgage-backed securities, a market worth more than $11 trillion. The largest groups are: depository institutions (32%), The Federal Reserve (23%), mutual funds (7%), money market funds (5%), pension funds and credit unions 5%, state and local governments (4%) insurance companies (3%), households and nonprofit organizations (2%), rest of the world (11%).

[5]See for example Benmelech and Dlugosz [2009] who describe a large data set of 3237 CLO's with an average of 7.3 tranches per issue. They explain how tranching is driven by investors' demand through the constraints those investors face. About 70% of the value in their data is concentrated in the most senior AAA tranches and the authors explain how credit ratings are computed.

[6]Another economic context where our results are applicable is crowdfunding: since May 2016, the Securities and Exchange Commission has allowed firms to issue debt (peer-to-peer lending) and equity securities to raise no more than $1,000,000 annually from non-professional investors. Our results can be used to derive the optimal menu of crowd funding rewards.

As previously mentioned, we provide a complete characterization of the optimal menu of offered securities and explain how it depends on the model's main features: the size of the financing need relative to available budgets, the relative risk aversions of all involved agents (issuer and investors), the relative frequency of investors with different risk appetites and their budget constraints. If the issuer's financing need cannot be fully covered by the least risk-averse (aggressive) investors, more risk-averse (conservative) investors must be attracted with less risky securities, yielding the sequential senior-junior security structure where each additional dollar of realized return is allocated to a unique class of investors, forming a "waterfall" structure. A higher financing need generally leads to more tranches being offered.[7] If the issuer is less risk averse than all types of investors, then the offered securities are debt contracts with different seniority and risk-return profiles while the issuer retains then an equity tranche. Finally, if the issuer's risk aversion is such that there are potential investors who are both more or less risk averse than the issuer, then the aggressive, least risk-averse investors buy the equity tranche.

Assuming the issuer is the least risk-averse party, we also find that if the issuer owns a stochastically better/safer asset (in terms of either a first-order or second-order stochastic shift) or if the issuer needs to raise less capital, then the optimally offered rate for senior debt decreases, and the issuer is better-off. Interestingly, in that case, the aggressive investors *always* get worse off. This is somewhat surprising since these investors are risk-averse and yet they prefer to invest in a security that is backed by a worse/more risky asset. This phenomenon is due to the effects of screening: aggressive investors earn information rents because they value the risky asset more than conservative investors. When the asset becomes less risky/better, the difference in valuations between the different types of investors decreases, and so does the information rent. This observation has interesting policy implications: market regulators often ban very risky assets in order to protect investors, but our results show that such restrictions may actually hurt risk tolerant investors. We also find that, if investors become more conservative, the issued securities become riskier so that the probability of not being able to serve the outstanding debt contracts increases.

The rest of the paper is organized as follows: In the remainder of this section, we survey the relevant literature. Section 2 presents the security design model with risk-averse and budget-constrained, heterogeneous investors. In Section 3, we describe and characterize feasible and incentive-compatible mechanisms in this setting. Opti-

[7]Cuchra and Jenkinson [2005] describe a varied set of 1605 securitized issues comprising 5161 separate tranches (on average 3.22 tranches per issue) that raised about \$1 trillion. Issues over \$1 billion in size have over 5 tranches on average, issues of \$100 million or less have 2 tranches on average. A similar finding appears in Vink and Thibeault [2008] who compare the number of tranches and level of subordination in 3467 structured issues (MBS, CDO, ABS).

mal security design via tranching is derived in Section 4. Section 5 presents several comparative statics results about changes in the optimal design when the underlying asset becomes stochastically better or safer, or less costly to implement. Section 6 offers several extensions to the basic model where we consider an issuer who is 1) takes into account the possibility of trading among investors after the initial issue, 2) is subject to moral hazard, 3) is herself risk-averse , 4) faces investors whose budgets are also private information and 5) faces more than two types of investors. Section 7 concludes.

1.1 Related Literature

The financial literature on security design is extensive. Accessible accounts of the role securitization played in the financial crises are given by Coval et al. [2009] and Hellwig [2009]. Gorton and Metrick [2013] and Allen and Barbalau [2022] provide comprehensive surveys of the practice and theory of securitization. We focus below on the aspects most relevant to our paper.

Classical security design models assume that the issuer is relatively more informed than the investors (see for example Leland and Pyle [1977], Myers and Majluf [1984], Nachman and Noe [1994], DeMarzo and Duffie [1999], Inostroza and Tsoy [2022], Malenko and Tsoy [2023]). Uninformed investors draw then inferences about the assets' merit from the contracts proposed by the informed issuer. These models focus on securities' sensitivity to the issuer's private information, and derive conditions under which debt dominates other securities. In an incomplete market model with noise traders, Boot and Thakor [1993] showed that the issuer's expected revenue is enhanced by selling both equity and debt rather than selling a single claim. Such a partition is profitable because it enables the decomposition of the cash flow into an information insensitive component and a sensitive component that is dependent on the seller's information. De Marzo [2005] also investigates how the asymmetry of information interacts with the ex-ante incentives to pool assets before securitization.[8]

Frank and Goyal [2003] and Fama and French [2005] criticized the above "pecking order" theory with its main driving force - the informational asymmetry between issuer and investors - and observe that firms issue much more equity than predicted by that theory (where it should be only the a "last resort" security retained by the issuer). Ospina and Uhlig [2018] compare ex-ante credit ratings of a large set of mortgage-backed securities with (post financial crisis) "ideal" ratings given the observed out-

[8]DeMarzo, Frankel, and Jin [2021] extend the "pecking order" theory by studying an issuer who holds multiple assets and who designs multiple securities before and/or after she becomes informed. It is optimal for the issuer to pool all her assets, and the issuer can wait and issue a single debt equity, or she can first tranche the pooled asset and sell those tranches whose seniority exceeds an information-sensitive threshold.

comes. They conclude that ex-ante ratings were relatively accurate (particularly on top AAA tranches) - the lemons problem was not that severe.

A smaller literature assumes that the outside investors rather than the issuer have superior information about the project's prospects[9]. Several papers following De Marzo et al [2005] study a model in which privately informed investors choose securities from a set ordered by steepness, rather than competing with securities designed by the seller as in Axelson's [2007] model.[10] Ball and Pekkarinen [2024] analyzed optimal selling mechanisms with royalties and costly verification. Figueroa and Inostroza [2023] showed that debt contract is the optimal screening contract that minimizes information rents of liquidity providers. In most of the above studies investors are risk-neutral, and the transaction occurs between the issuer and a unique winner, who is the only one to make a payment. In particular, tranching based on risk aversion does not play a role in the obtained results.[11] Rostek and Yoon [2021] study the design of securities in a multi-asset model where assets are sold in separate, uniform-price auctions. They find that in markets with large traders, a limited number of synthetic products, customized to align with traders' desired risk profiles, generally outperform mutual funds in welfare terms.

Somewhat surprisingly, investors' risk aversion is not a standard feature in models of security design. For instance, it is absent in the classical models that derive the optimality of debt contracts by appealing to costly verification, bankruptcy penalties or moral hazard[12]. This is mostly due to the high technical difficulty of such an analysis within the framework of expected utility. Hellwig [2001] shows that the optimality of debt fails if the borrower is risk-averse. In that case, the optimal contracts have a complex mathematical form. Allen and Gale [1989] and Malamud et al. [2010] study optimal security design and risk sharing with risk-averse investors. In particular, Allen and Gale show that, in a general equilibrium model with expected utility, neither debt nor equity are optimal securities. Thus, their model and analysis cannot explain the emergence of the standard securities that are observed in practice.[13] Gershkov et al [2023] focus on classical monopolistic insurance with dual risk-averse agents, and derive

[9]Axelson [2007] argues that this fits well situations where start-up companies seek to raise funding from professional investors.

[10]Subsequent work has extended the framework of De Marzo et al in order to allow for adverse selection (Che and Kim [2010]), competing sellers (Gorbenko and Malenko [2011]), endogenous entry of bidders (Sogo et al. [2016]) and negative externalities (Hernandez-Chanto and Fioriti [2019]).

[11]Abhishek et al. [2015] and Fioriti and Hernandez-Chanto [2022] allow for risk-averse bidders. In contrasts to our result below, the later paper finds that, for sufficiently risk-averse bidders, it is efficient to only sell call options.

[12]See for example Townsend [1979], Gale and Hellwig [1985], Diamond [1984] and Innes [1990].

[13]This is related to the classical insights about Pareto efficient risk sharing under EU preferences, due to Borch [1962] and Wilson [1968]: explicit solutions can be computed only in special cases, and do not correspond to tranching.

the optimal menu of contracts for an insurer that maximizes revenue: in general these menus offer *layer* insurance where each additional dollar of potential loss is either fully retained by the insuree or fully passed to the insurer.[14] Contrasting the present framework, the insurance setting involves an uninformed insurer with unlimited funds who buys part of the risk faced by an agent (with known risk preferences) who is privately informed about the distribution of accidents.

A number of papers study security design problems with risk-neutral agents endowed with heterogeneous beliefs (see, for example, Garmaise [2001], Broer [2018], Simsek [2013], Ellis et al. [2022], Ortner and Schmalz [2019] and Luo and Yang [2023]). Some of these models a-priori assume tranching even if it is not optimal (e.g. Broer, [2018]), while others derive optimal tranching structures consisting of securities that are, in general, different from debt or equity. Ellis et al. [2022] study a competitive equilibrium model where both issuer and investors are risk-neutral price-takers. In equilibrium, the possible realizations of future returns are partitioned into intervals: each interval is "assigned" to a single investor with the maximal willingness to pay for that interval. Hence, this equilibrium can be also be implemented via tranching where each trader may buy more than one tranche. Allowing for diverse risk preferences while assuming that beliefs are homogenous does not generate tranching in that model. Moreover, in that case the equilibrium security design need not allocate risk to those most willing to bear it. Ortner and Schmalz [2019] analyze a security design model with risk neutral agents and without the main frictions present here (the investors' budget constraints and the issuer's need to raise cash). The structure of the optimally issued securities are determined by the relations (conditional on the state of the world) between the optimism/pessimism of issuer vs investors and among investors. In our model the basic waterfall structure consisting of seniority-ordered standard debt tranches and one standard equity tranche is optimal independently of whether the issuer is more or less risk averse than some or all investors - what changes is the allocation of the equity tranche.

Finally, we note that the interesting class of models with heterogenous beliefs is not fully consistent with an optimal mechanism design analysis: when it is common knowledge that market participants have diverse beliefs (i.e., when they "agree to disagree"), the issuer can arbitrage the differences in beliefs by organizing structured trades among the investors, hereby extracting the whole available surplus. In other words, in order to rationalize the use of standard securitization policies, it is necessary to impose exogenous restrictions on the class of feasible mechanisms.[15] As demon-

[14]Under some additional regularity assumptions, optimal contracts take the form of menus of different deductibles up to full insurance, or menus of full insurance up to different coverage limits.

[15]A possible interpretation of our model is that agents have distorted beliefs that overweight more adverse events, leading to non-linear probability weighting. But, this interpretation is not straightfor-

strated in Section 6.1, our results are also robust to allowing trading among agents after the initial issuance - a feature of many financial markets. In contrast, incorporating trading into a mechanism design model with heterogeneous beliefs may be challenging for the reasons discussed above.

2 The Model

A seller/issuer (she) has a project or asset that generates a random return with outcomes in the interval $X = [0, \bar{x}] \subseteq \mathbb{R}_+$. The project's return x is governed by the distribution $H : X \to [0, 1]$. The seller has no cash, and needs to raise funds of $c \in (0, 1)$ in order to finance the project.[16]

Risk Preferences and Budgets of Investors There is a unit mass of potential, risk-averse investors/buyers/agents (he). Each one of them is described by a limited budget that is normalized here to 1, and a preference over lotteries.

Let $\mathcal{V}$ be the set of random variables with outcomes in the interval $X = [0, \bar{x}]$, defined on a given non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A functional $\rho : \mathcal{V} \to \mathbb{R}$ is a distortion risk-measure if it satisfies the following properties (see Wang [1996]):

(i) $\rho(v + w) = \rho(v) + \rho(w)$ if $v, w \in \mathcal{V}$ are comonotonic.

(ii) $\rho(1) = 1$.

(iii) ρ is decreasing in first-order stochastic dominance

A distortion risk-measure is *coherent* if it is subadditive:

(iv) $\rho(v + w) \leq \rho(v) + \rho(w)$ for all $v, w \in \mathcal{V}$.

Subadditivity is an appealing property because it ensures that diversification reduces risk, reflecting the benefit of spreading investments across multiple assets. Part (i) states that there is no benefit of diversification when investing in perfectly correlated assets. Throughout, we assume that the risk preference of each investor is consistent with a coherent *distortion* risk measure – these are also called *spectral* risk measures.

The investors's risk preferences over risks are consistent with a distortion risk measure if and only if there exists an increasing function $g : [0, 1] \to [0, 1]$ with $g(0) = 0$

ward since investors' beliefs would endogenously depend here on the underlying mechanism, i.e., on the issued securities. This is due to the rank-dependent nature of the used distortion risk measures

[16]Our model can trivially be extended to allow the issuer to choose different investment levels c with associated return distributions H_c. We characterize the cost-minimizing securities needed to raise any fixed c, and the resulting profit of the issuer. The optimal investment level can be obtained by maximizing over c and our characterization of optimal securities holds unchanged.

and $g(1) = 1$ such that the certainty equivalent of the investor is given by[17]

$$\mathcal{U}_g(v) = \int_0^{\bar{x}} g(\mathbb{P}[v > s])ds\,.$$

A distortion risk measure is *coherent* if and only if g is convex; which is also equivalent to risk aversion in the standard sense of aversion to mean preserving spreads. For $H_v(s) = \mathbb{P}[v \leq s]$ and g absolutely continuous integration by parts yields[18]

$$\mathcal{U}_g(v) = \int_0^{\bar{x}} g'(\mathbb{P}[v > s])\, s\, dH_v(s)\,.$$

The above certainty equivalent modifies the standard expectation operator $\mathbb{E}_H[v] = \int_0^{\bar{x}} s\, dH_v(s)$ by distorting each outcome s by the weight $g'(\mathbb{P}[v > s])$ that is decreasing if g is convex, i.e., if the agent is risk-averse. Thus, risk aversion is created here by having higher weights on less favorable outcomes. The ubiquitous channel that creates risk aversion - decreasing marginal utility of money - is absent and the marginal utility of money is constant.

Private Information Each investor's risk preferences is his private information, and described by a type θ that determines the distortion function g_θ. We assume for simplicity that there are two types of investors: l types (*conservative* investors) with low risk-tolerance and h types (*aggressive* investors) with high risk-tolerance $\theta \in \Theta = \{l, h\}$. Each type occurs with probability $f_l, f_h > 0$, respectively, such that $f_h + f_l = 1$. In Section 6.5 we extend our results to more than two types.

We assume that the investors are risk averse and that their risk attitudes are ordered: g_l is a convex transformation of g_h meaning that conservative type l investors are more risk-averse than the aggressive type h investors.[19] It is noteworthy that this also implies here that $g_l(p) \leq g_h(p)$ for all $p \in [0, 1]$, with strict inequality holding for some p. We first assume, for simplicity, that the issuer is risk neutral. In Section 6.3 we will discuss the extension to a risk-averse issuer.

Finally, to rule out trivial cases, we assume that it is technically feasible to raise the necessary funds c from investors. Concretely, we assume that the conservative investors – who require a higher premium in order to purchase risk - value the project higher than its cost, i.e. $\mathcal{U}_{g_l}(v) > c$. As investors are risk-averse, this condition also implies that the project's expected return is at least c.

Before concluding this section, we note that having the same budget for both types

[17] See Theorem 4 in Wang [1996].

[18] The integral here is in the Lebesgue-Stieltjes sense.

[19] For comparisons of risk aversion attitudes and their relation to the convexity of probability distortion functions and the transformations thereof, see Yaari [1986].

is for illustrative convenience only. Our results generalize in a straightforward manner to the case where budgets are heterogeneous and are the agents' private information (see Section 6.4 for details).

2.1 Examples of Distortion Risk Measures

Special cases of distortion risk measures are:

Value at Risk Value at risk (VaR) is a commonly used risk-measure, defined as $VaR(v) = H_v^{-1}(\alpha)$. It represents the threshold s where the likelihood of achieving a return greater than s is $1-\alpha$. This risk measure corresponds to the distortion

$$g^{\alpha}_{VaR}(p) = \begin{cases} 1 & \text{if } p \geq 1-\alpha \\ 0 & \text{else} \end{cases} .$$

Value at risk is not coherent, and hence its use is not anymore recommended by the Basel framework.

Expected Shortfall The main alternative used in practice, and now explicitly mandated by the Basel Framework is *expected shortfall* (also called *conditional VaR*). It is a coherent risk measure that refines VaR by considering the expected return conditional on it lying below the level that is exceeded with probability $1-\alpha$

$$ES(v) = \mathbb{E}[v | v \leq VaR(v)] .$$

Expected shortfall corresponds to the distortion

$$g^{\alpha}_{ES}(p) = \frac{1}{\alpha} \max\{p - (1-\alpha), 0\} .$$

Exponential Distortion Risk Measure Another popular ordered class of coherent risk measures, parametrized by $\alpha \in [0,1]$, corresponds to the family of *exponential distortions*:

$$g^{\alpha}(p) = \frac{e^{-\alpha(1-p)} - e^{-\alpha}}{1 - e^{-\alpha}} .$$

Remark: Even if the investor is risk neutral but the set of feasible investments is regulatorily constrained by an upper bound on the risk-measure associated with the investment, a Lagrange multiplier approach induces the maximization of a weighted sum of the expected value and the associated security equivalent, e.g. $(1-\lambda)\mathbb{E}[v] +$

$\lambda \mathcal{U}_{\tilde{g}}(v)$. The corresponding distortion function is given by

$$g_\lambda(p) = (1-\lambda)\, p + \lambda\, \tilde{g}(p)\,.$$

Note, that as the weight λ on the risk-measure increases, g_λ becomes more convex, and the investor becomes more risk averse. Thus, investors who are more constrained in terms of how much risk (measured by a spectral risk-measure) they can accept are more risk-averse.

2.2 Alternative Interpretation

Alternatively, our investors can be viewed as risk-averse individuals with non-expected utility functions, whose risk preference arise from non-linear probability weighting. The preferences induced by spectral risk measures correspond coincide with the so-called *dual* risk-averse utility functionals from decision theory (Yaari [1987]).[20] Dual utility induces first order risk-aversion which makes it appealing for many applied contexts, such as insurance decisions or investment decisions. We provide a more detailed discussion of the empirical evidence on risk preferences in financial markets in Supplemental Appendix C.

Another well-known example in behavioral economics is the class of *reference-dependent loss-averse* preferences with linear local utility, studied by Kőszegi and Rabin [2006] and Masatlioglu and Raymond [2016], among others. These preferences correspond to the distortion function:

$$g^k(p) = kp^2 + (1-k)p\,,$$

where $k \in [0,1]$ captures the degree of risk aversion. For $k=0$, preferences reduce to risk neutrality, while $k=1$ represents the highest degree of risk aversion in this class.

3 Mechanisms: Menus of Securities

We restrict attention to deterministic direct mechanisms. Stochastic securities are rarely, if ever, used in practice - it is very hard to credibly commit to the announced randomizations - and we abstract from them here.

Since we have a continuum of agents, we look at mechanisms $(R_\theta, t_\theta)_{\theta\in\Theta}$ consisting of a menu of *asset-backed securities* R_l, R_h and their prices t_l, t_h. For each θ, $R_\theta(x)$ is the payoff of the security R_θ if the underlying assets return equals x and $t_\theta \geq 0$ is

[20] For a detailed exposition on risk measures and their connections to axiomatic, non-expected utility, see Arztner et al. [1999] or Rüschendorf [2013], Chapter 7.

the *price* of this security. We restrict attention to monotonic contracts having non-negative returns. That is, for any θ and x, $R_\theta(x) \geq 0$ and $R_\theta(x)$ is non-decreasing in x. We also assume that, for each θ, the payoff of the security $R_\theta(x)$ is absolutely continuous in the project's return, and we denote by $R'_\theta(x)$ its generalized derivative. In addition, the promised return of all offered asset-backed securities cannot exceed the value of the underlying asset. That is, for any $x \in X$, the following feasibility constraint must hold:

$$\sum_{\theta \in \{l,h\}} f_\theta \, R_\theta(x) \leq x.$$

It directly follows that $R_\theta(0) = 0$ for all θ.

Fix any mechanism $(R_\theta, t_\theta)_{\theta \in \Theta}$. An agent with type θ who reports to be of type θ' obtains utility

$$U(\theta, \theta') = -t_{\theta'} + \int_X R'_{\theta'}(x) g_\theta (1 - H(x)) dx.$$

Remark: Payments made to investors in the context of asset-backed securities are exclusively sourced from the asset's returns. Alternatively, one can extend the model by allowing the issuer to add the capital collected from investors in order to increase some of the payments. This modifies the feasibility constraint to:

$$\sum_{\theta \in \{l,h\}} f_\theta \, R_\theta(x) \leq x + \sum_\theta f_\theta t_\theta - c.$$

However, as demonstrated in the Supplemental Appendix A, it is never strictly optimal for the issuer to adopt this re-investment approach. It is thus without loss of generality to focus on purely asset-backed securities.

3.1 Implementable Mechanisms

As the conservative l type of investor assigns a strictly higher value to the project than its cost, the issuer can always offer a single security that will be bought by all types, and that has a price greater than c. Doing so yields a strictly positive profit, so it cannot be optimal not to finance the project. This yields that, in any optimal menu, the collected revenue exceeds the financing cost c

$$f_h t_h + f_l t_l \geq c. \tag{FC}$$

Therefore, we only consider below mechanisms in which the project is successfully financed, and thus (FC) needs to hold. For a type θ agent not to deviate and claim to be of type θ', it needs to hold that $U(\theta, \theta) \geq U(\theta, \theta')$. This is the same as having the difference in the security equivalents of the assets exceed the difference in their prices:

$$\int_X \left[R'_\theta(x) - R'_{\theta'}(x)\right] g_\theta(1 - H(x))dx \geq t_\theta - t_{\theta'}\,. \tag{IC-θ}$$

Similarly, in order to make a type θ agent purchase the security offered to him instead of pursuing an outside option that is normalized here to yield zero utility (e.g., acquiring a risk-free government bond with a fixed interest rate), it must be the case that

$$\int_X R'_\theta(x) g_\theta(1 - H(x))dx \geq t_\theta\,. \tag{IR-θ}$$

The feasibility constraint requires that the promised payments from the asset cannot exceed the value of the asset: $f_l\, R_l(x) + f_h\, R_h(x) \leq x$ for each $x \in X$. That is equivalent to $R_\theta(0) = 0$ for $\theta \in \Theta$ and for all $x > 0$

$$\int_0^x [f_l R'_l(z) + f_h R'_h(z)]dz \leq x \tag{Feasibility}$$

In addition, recall that we require that the return of the security is increasing in the return of the underlying asset

$$R'_\theta(x) \geq 0 \tag{M}$$

for $\theta \in \Theta$ and for all x, and that each type θ has a limited budget of 1:

$$t_\theta \leq 1. \tag{BC}$$

4 Optimal Security Design

The issuer's profit when financing the project via the mechanism $(R_\theta(x), t_\theta)_{\theta \in \Theta}$, is given by:

$$\mathbb{E}_H[x] - \int_X \left[f_h R'_h(x) + f_l R'_l(x)\right](1 - H(x))dx + \left[f_h t_h + f_l t_l - c\right].$$

Our first Lemma demonstrates that the issuer never wants to raise more than c: raising funds that are not strictly needed is too costly in terms of the foregone returns that can be obtained by retaining a higher portion of the underlying asset. This happens because all investors demand here a risk premium in order to acquire risk, while the seller is risk-neutral.

Lemma 1 *Fix any optimal menu $(R_\theta, t_\theta)_{\theta \in \Theta}$. It must hold that*

$$f_h t_h + f_l t_l = c. \tag{FC'}$$

By the above Lemma, and since the asset's expected return $\mathbb{E}_H[x]$ is fixed, the seller's objective function reduces to:

$$\min_{(R_\theta(x),t_\theta)_{\theta\in\Theta}} \left\{ \int_X \left[f_h R_h'(x) + f_l R_l'(x)\right](1 - H(x))dx \right\}$$

subject to constraints (IR), (IC), (Feasibility), (M), (BC) and (FC'). In words, the issuer wants to minimize the loss of potential cash-flow from the asset caused by the sale of securities to investors, subject to the constraints that she needs to raise a sum c from them, and that the mechanism is implementable.

We next derive the optimal security design, distinguishing between two cases. In the first case, the project can be entirely financed by selling securities solely to aggressive investors. The solution to the first case offers a building block for the second, more complex case, where it is necessary to design securities for both types of investors.

Investors' Preferences over Securities For later use, we first recall a fundamental result from the insurance literature (see, for example, Van Heerwaarden et al.[1989]), adapted to our security design framework. Let $\mathcal{R}_\gamma$ denote the set of all feasible, monotonic securities R that yield the same expected payment $\gamma = \mathbb{E}[R(x)]$. Define x^d by solving $\mathbb{E}[\min\{x, x^d\}] = \gamma$, and note that $R^\gamma_{debt}(x) = \min\{x, x^d\}$ represents a *debt* contract in $\mathcal{R}_\gamma$.

Theorem 1 *Consider any security $R \in \mathcal{R}_\gamma$. Then $R(x) \preceq R^\gamma_{debt}(x)$ where $\preceq$ denotes second-order stochastic dominance.*

In other words, among all monotonic securities with a given expected return, debt is the least variable security and is therefore the best choice for **any** risk-averse investor.

4.1 Aggressive Investors are Sufficient to Finance the Project

Suppose first that the project can be financed by raising money only from the high types, i.e. $c \leq f_h \leq 1$. In this case, the most efficient way of raising the necessary funds is to offer securities only to the aggressive investors, so that it is optimal to set $t_h = \frac{c}{f_h}$ and $t_l = 0$. An alternative interpretation of this basic scenario is one in which a single investor, with a sufficiently large budget, finances the entire project, as in a specially tailored, single-tranche CDO.

The resulting maximization problem for the issuer becomes:

$$
\begin{aligned}
\text{(Problem P)} \quad & \min_{R_h} \left\{ f_h \int_X R_h'(x)(1 - H(x))dx \right\} \\
\text{s.t.} \; & \int_X R_h'(x) g_h(1 - H(x))dx = \frac{c}{f_h} \\
& R_h'(x) \geq 0, && \forall x \in X \\
& f_h \int_0^x R_h'(z)dz \leq x, && \forall x \in X \\
& R_h(0) = 0
\end{aligned}
$$

The first equality in the above set of constraints is the binding participation constraint for the aggressive types who exhaust their budget. This is an *isoperimetric* constraint. The second inequality is the *monotonicity* constraint. The last two constraints ensure that the payouts to the agents do not exceed the return of the security (i.e. the feasibility) where the second last constraint is a *majorization* constraint, and the last constraint is a *boundary condition*.

The result below shows that the optimal security in this case is a debt contract targeting the aggressive investors. The debt's interest rate is chosen so that the aggressive type's participation constraint is binding.

Proposition 1 *Let x^* denote the solution to*

$$
\int_0^{x^*} g_h(1 - H(z))dz = c
$$

and note that $c < x^* < \bar{x}$.[21] *The optimal security is given by*

$$
R_h^*(x) = \frac{1}{f_h} \min\{x, x^*\}
$$

and by $t_h^* = c/f_h$. That is, a debt contract with interest rate $x^*/c - 1 > 0$ is optimal.

The convexity of g_h is equivalent here to the assumption that the function $z \mapsto \frac{z}{g_h(z)}$ is decreasing, or that the function $x \mapsto \frac{1-H(x)}{g_h(1-H(x))}$ is increasing. Intuitively, when this last condition holds, it is beneficial for the issuer to make $R_h'(x)$, the security's slope, as large as the majorization constraint allows for small realizations of the asset's return x, and as small as possible for larger realizations obtained as soon as the isoperimetric constraint is satisfied. Instead of a direct proof based on this observation, we utilize

[21] Such x^* exists because, by assumption $\int_X g_h(1-H(x))dx \geq c$ so that the project can be financed at all.

Theorem 1: among securities with a given cost, a debt contract yields the highest utility to a risk-averse agent. Equivalently, to secure a fixed utility level for such an agent—as required by (IR-h)—the issuer's most cost-effective strategy is to offer a debt contract. Notably, this conclusion holds for any risk-averse agent, regardless of whether their preferences align with EU or non-EU models.

Observation: The above proposition shows how the optimal security's interest rate depends on the risk-aversion of the aggressive investors, on the project's return profile, and on the financing cost c. It includes several very intuitive predictions that can be tested empirically: ceteris paribus, a better distribution of returns H (in the sense of first order stochastic dominance) yields a lower interest rate; a higher degree of risk-aversion yields a higher interest rate; a higher financing need c also yields a higher interest rate.[22]

4.2 Both Types of Investor are Needed to Finance the Project

Consider now the case where both types of investor are needed to finance the project, i.e. $c > f_h > 0$. For this setting, the participation constraint of a low type, (IR-l), must be binding. If not, then the seller could increase her profit by extracting higher payments from both types. Analogously, the incentive constraint of a high type, (IC-h) must also bind.

To derive the optimal mechanism, we consider a relaxed problem where we ignore the IC constraint of the low type and the participation constraint of the high type, and later verify that the ignored constraints are not binding. We then establish that, in any solution, the issuer must completely exhaust the budget of all agents with high risk tolerance in order to finance the project. As it is never optimal to raise more money than the cost of the project, this determines the amount of money raised from both types of investors. We then establish that the optimization problem can be decoupled across investor types: one can first maximize over the security offered to the high type (fixing the low type security) and then over the security offered to the low type (fixing the high type's security). As each of these two problems have a single type of investor, an argument similar to the one of Proposition 1 yields that both offered securities are debt contracts.

[22]To see the last point, vary c and let $x(c)$ be a solution to $\int_0^{x(c)} g_h(1 - H(z))dz = c$. Taking the derivative with respect to c twice in the above expression yields that x is convex and hence the corresponding interest rate, $\frac{x(c)}{c} - 1$, is increasing in the financing need c.

Theorem 2 *Suppose that $c > f_h$ and let x_l^*, x_h^* denote the solutions to:*

$$\begin{aligned} \int_0^{x_l^*} g_l(1 - H(x))dx &= c - f_h \\ \frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1 - H(x))dx &= \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H(x))dx + \frac{1-c}{f_l}, \end{aligned}$$

respectively. The optimal menu of securities is given by $t_l^* = \frac{c-f_h}{f_l}$, $t_h^* = 1$, and by

$$R_l^*(x) = \begin{cases} \frac{x}{f_l} & \text{for } x \leq x_l^* \\ \frac{x_l^*}{f_l} & \text{otherwise} \end{cases}.$$

$$R_h^*(x) = \begin{cases} 0 & \text{for } x \leq x_l^* \\ \frac{x - x_l^*}{f_h} & \text{for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^* - x_l^*}{f_h}, & \text{otherwise} \end{cases}.$$

The proof of the theorem can be found in the Appendix. It establishes that the optimal mechanism is a menu of two contracts:

1. *Senior debt* with interest rate

$$\frac{\frac{x_l^*}{f_l}}{t_l^*} - 1 = \frac{\frac{x_l^*}{f_l}}{\frac{c-f_h}{f_l}} - 1 = \frac{x_l^*}{c - f_h} - 1$$

   that is determined by the participation constraint of the conservative investors (i.e. (IR-l)).

2. *Junior debt* with interest rate

$$\frac{\frac{x_h^* - x_l^*}{f_h}}{t_h^*} - 1 = \frac{x_h^* - x_l^*}{f_h} - 1,$$

   that is determined by the incentive compatibility constraint of the aggressive investors (i.e. (IC-h)).

Furthermore, the amount of money that any agent can invest in senior debt is limited to $\frac{c-f_h}{f_l}$.

To gain some intuition on why this "waterfall" structure is optimal, note first that a similar structure of securities (i.e., senior/ junior debt and equity) is also optimal under complete information. With complete information, the designer only needs to induce an efficient risk sharing among agents (subject to participation constraints). As the investors' risk preferences are ordered, the efficient way to share risk is to allocate the safest possible asset (i.e., senior debt) to the most risk-averse investor type, to allocate the second safest possible asset (i.e., junior debt) to the more risk tolerant investors, and to keep the remaining, most risky asset (i.e., equity). In this case, the interest rate for the senior debt is still determined by the (IR-l) constraint, and it is equal to that in the incomplete information setting. However, under complete information, the interest rate for junior debt is determined by the (IR-h) constraint, and it is lower than that under incomplete information where it is determined by the (IC-h) constraint.

The above optimal allocation presents an atypical structure compared to standard screening models where the allocation of the low type is distorted downwards from its efficient level in order to reduce the information rent extracted by the high type. This is **not** the case here since conservative investors (low types) obtain exactly the same product and pay the same price as in the complete information benchmark. By contrast, the interest rate offered to the aggressive investors (high types) is **distorted upwards** to meet their incentive compatibility constraint. This happens because the efficient risk allocation to the low types - senior debt with the minimal interest rate to guarantee their participation - also minimizes the high type's information rent. Since the high type earns a rent because he values the risky asset more than the low type, the information rent is minimized when the low type receives the safest asset, i.e., in an allocation that coincides with the solution to the complete information benchmark. Thus, the most cost-effective way to provide the high type the required information rent is to maintain the waterfall structure that is optimal under complete information while increasing the interest rate for junior debt. We will further illustrate this point in Section 6.

Remark (Expected Utility) : Even under complete information, a "waterfall" structure is not optimal if investors have expected utility preferences. This happens because, in such cases, the marginal utility of money is not constant, leading to a much more complex risk-sharing problem (see, for example, Allen and Gale [1989]).

Remark (Inefficiency of Investment): Suppose that the seller can choose any investment level $c > 0$, each associated with a return distribution H_c. It follows from Theorem 2 that if both investor types are needed to raise funds of c, then in addition to compensating all investors for risk taking, the agressive investors earn an information rent when the investment level c is chosen. Thus, as the issuer does not fully internalize the benefits of increasing the investment level she will in general **underinvest** in the project.

5 Comparative Statics

In this section we present several comparative statics results. These results are markedly different from those obtained in the same model under complete information. We focus here on the more interesting case where both types of investor are needed for financing the project, e.g. assume $f_h < c$. We first investigate the effects of having a better/safer asset in the sense of a first order stochastic dominance (FOSD) or second order stochastic dominance(SOSD) shift, respectively. We then discuss the effect of a decrease in the financing cost.

To simplify notation let

$$i_l^x = \frac{x_l^*}{c - f_h} - 1 \qquad\qquad i_h^x = \frac{x_h^* - x_l^*}{f_h} - 1$$

denote the interest rates offered by the issuer to the conservative and aggressive investors, respectively, in the optimal menu of asset backed securities.

5.1 Who Prefers a Better or a Safer Asset?

Our main finding here is that the issuer always prefers a stochastically better/safer asset, while the aggressive investors always prefer a worse/riskier asset. Recall that H denotes the distribution of returns of the asset.

Proposition 2 *Either a SOSD or FOSD improvement of H results in:*

1. *A decrease in the interest rate i_l for conservative investors.*
2. *A decrease in the expected cost of financing $\int_0^{x_h^*}(1 - H(z))dz$.*
3. *A decrease of the surplus $\int_{x_l^*}^{x_h^*} g_h(1 - H(z))dz - 1$ obtained by the aggressive investors.*

Furthermore, an FOSD shift of H leads to a decrease in the interest i_h obtained by aggressive investors.

In the optimal mechanism conservative investors are indifferent between purchasing and not purchasing the asset . With a stochastically better/safer asset, the issuer offers a lower interest rate to conservative investors, gives up a smaller share of the asset, and leaves less information rent to the aggressive investors. With a stochastically better asset, the interest rate offered to aggressive investors will also decrease. Intuitively, the comparative advantage of aggressive investors lies in their higher tolerance for risk. If the asset becomes less risky this advantage decreases, and the rents of aggressive investors decrease as they can be more easily substituted by conservative investors.

5.2 The Project's Cost

We find that, as the project becomes less costly to implement, both the offered interest rates and the financing cost decrease. Conservative investors are indifferent, but the aggressive investors are worse-off.

Proposition 3 *In the optimal contract that finances asset x and raises c, the following hold:*

1. *Both interest rates, i_l and i_h, increase in c;*
2. *Aggressive investors prefer to finance costlier projects since their utility $\mathcal{U}_{g_h}(R_h(x)) - t_h$ increases in c.*

As the cost of the project increases, the issuer needs to forgo a larger share of the security to raise sufficient funds. Consequently, the riskiness of both senior and junior debt rises, necessitating higher interest rates. Moreover, as the asset sold to the conservative investors becomes riskier, the aggressive investors can capture a higher information rent.

6 Extensions

In this section we offer several extensions to our basic model. In each of these extensions, we introduce only one modification at a time, keeping all other aspects identical to the baseline model, and we focus on the more interesting case where multiple investor types are needed for financing. In particular we consider:

1. the security design problem where the issuer cannot impose purchase limits;
2. security design where the issuer can also undertake an action that affects the asset's return (moral hazard);
3. security design by a risk averse issuer;
4. the case where investors' budgets are also their private information; and finally
5. the generalization of our basic framework to more than two types of investors.

A central result that emerges across these extensions is that the waterfall structure of subordination remains optimal under a wide range of modifications to the original model.

6.1 No-Purchasing Limits

We characterize here the optimal mechanism for the case where purchasing limits, as featured in the benchmark model, cannot be imposed. While above we explored the optimal mechanism design problem where the issuer fully controls both the allocation and monetary transfer of every type, here we consider a simpler and more prevalent selling procedure where every investor decides how many units of each security to acquire. The main change lies in the incentive constraint of the aggressive investor type who must now be deterred from buying multiple units of the security intended for the conservative types. This influences the offered interest rates, but does not alter the waterfall structure of the optimal menu.

For an alternative motivation, note that if agents can trade among themselves after the initial issue, aggressive, less risk-averse investors may be able to purchase securities from the conservative, more risk-averse investors at a mutually beneficial price. Foreseeing this possibility, the aggressive investors will refrain from buying at the initial issue, causing a loss of revenue to the issuer. The incentive constraint imposed in this section is precisely constructed in order to prevent this possibility: there will be no incentives for trading among the various investors after the initial issue. We note that, even under complete information about the investors' risk types, the issuer cannot effectively use it in order to eliminate the agents' information rent: conservative investors could purchase additional units of securities with the intent of resale to aggressive investors, thereby undermining the issuer's ability to enforce the intended allocation.

In any implementable mechanism where both types are needed to finance the project, an aggressive investor derives strictly positive utility from pretending to be a conservative type. As our risk preferences are homogenous, the aggressive type's best deviation without purchase limits is to purchase $\frac{1}{t_l} > 1$ units of R_l. (IR-l) is then still given by

$$\int_X R_l'(x) g_l (1 - H(x)) dx = \frac{c - f_h}{f_l} \, ,$$

but the incentive constraint of the high type (IC-h) changes to

$$\int_X \left[R_h'(x) - \frac{f_l}{c - f_h} R_l'(x) \right] g_h (1 - H(x)) dx = t_h - t_l \times \frac{1}{t_l} = 0.$$

Theorem 3 *Suppose that $c > f_h$ and let $\hat{x}_l$, $\hat{x}_h$ denote the solutions to:*

$$\int_0^{\hat{x}_l} g_l (1 - H(x)) dx = c - f_h$$

and

$$\frac{1}{f_h} \int_{\hat{x}_l}^{\hat{x}_h} g_h (1 - H(x)) dx = \frac{1}{c - f_h} \int_0^{\hat{x}_l} g_h (1 - H(x)) dx,$$

respectively. The optimal menu of securities is given by $\hat{t}_l = \frac{c-f_h}{f_l}$, $\hat{t}_h = 1$, *and*

$$\hat{R}_l(x) = \begin{cases} \frac{x}{f_l}, & \textit{for } x \le \hat{x}_l \\ \frac{\hat{x}_l}{f_l}, & \textit{otherwise} \end{cases}$$

and

$$\hat{R}_h(x) = \begin{cases} 0, & \textit{for } x \le \hat{x}_l \\ \frac{x-\hat{x}_l}{f_h}, & \textit{for } \hat{x}_l \le x \le \hat{x}_h \\ \frac{\hat{x}_h-\hat{x}_l}{f_h}, & \textit{otherwise} \end{cases}$$

The above theorem establishes that the optimal mechanism is still a menu of two contracts: one of them senior debt with interest rate $\frac{\hat{x}_l}{c-f_h} - 1$, and the other one junior debt with interest rate $\frac{\hat{x}_h-\hat{x}_l}{f_h} - 1$. Relative to the benchmark model with a purchasing limit, the new menu offers the same interest rate for senior debt holders, but a higher interest rate for the junior debt holders. This happens because the more aggressive investors can now earn more by deviating and buying $\frac{1}{t_l}$ units of senior debt. Thus, in order to ensure incentive compatibility, the interest rate offered for the junior debt must increase.

Remark (Interest Rates without Purchase Limits): It is intuitive that, in the present framework where investors are not constrained in their purchases, the interest rate for junior debt should always be higher than the one for senior debt and we establish in Lemma **??** in the Supplementary Appendix that this is indeed the case.

Remark (Risk-Aversion and Debt Default Probability): The above result offers many intuitive implications that can be empirically tested. For example, in the aftermath of the financial crisis of 2007-2008, it has been argued that investors became more risk-averse, with possible negative consequences on the viability of tranched CDO's (collateralized debt obligations).[23] Our model makes clear predictions about how the optimal securities change in such a case.

Let us assume, for example, that the conservative investors become even more risk-averse, and thus use a more convex distortion $\tilde{g}_l$. It directly follows that $\tilde{g}_l(p) \le g_l(p)$ for any $p \in [0, 1]$. To meet the (IR-l) constraint, the new relevant cutoff $\tilde{x}_l$ that solves

$$\int_0^{\tilde{x}_l} g_l(1 - H(x))dx = c - f_h,$$

must exceed $\hat{x}_l$. This implies the interest rate, given by $\frac{\tilde{x}_l}{c-f_h} - 1$, offered to these

[23] "What happened as a result of the crisis was that investors started asking more for an AAA tranche over similarly rated corporate bonds. Risk aversion increased, causing CDOs to fail." Jon Gregory, Solum Financials, cited in Sherif, N. (2014). The risk of risk aversion. *Risk*, **69**.

(more) conservative investors increases. This change also has implications for the optimal security $\tilde{R}_h^*$ presented to aggressive investors, whose characteristics have not changed. Given that (IC-h)

$$\int_{\tilde{x}_l}^{\tilde{x}_h} g_h(1 - H(x))dx = \frac{f_h}{c - f_h} \int_0^{\tilde{x}_l} g_h(1 - H(x))dx$$

remains binding, the relevant new cutoff $\tilde{x}_h$, and thus also the interest rates offered to the aggressive investors, must increase. The rise in interest rates, without a corresponding shift in the asset's return distribution, indicates an increased probability of failing to fulfill the obligations of the outstanding debt contracts. This finding aligns with the observations by Ospina and Uhlig [2018], who pointed out that mortgage-backed securities issued around the 2007-2008 financial crisis — a time when investors had become notably more risk-averse — were in fact more susceptible to defaults compared to those from earlier periods.[24]

6.2 Moral Hazard and "Skin in the Game"

We next consider a variant of the basic design problem where the securities' issuer needs to exert effort in order to (stochastically) increase the asset's returns. Chemla and Henessey [2014] and Malamud et al [2013], among others, analyze the issuer's incentives to affect the asset's returns, modeled as a moral hazard problem. In contrast to those authors, we do not assume a specific security design: we do derive the optimal structure of securities taking into account the moral hazard problem.

We maintain the assumption that the issuer needs to raise $c > 0$ to finance the project, has no funds, and is protected by limited liability. In this subsection and the next one, we restrict attention to *doubly monotonic* securities. A menu of securities, $(R_\theta)_{\theta=l,h}$ is doubly monotonic if, in addition to R_θ being monotonic for $\theta \in \{l, h\}$, the function

$$R(x) = x - \sum_{\theta=l,h} f_\theta R_\theta(x)$$

is also monotonic, i.e., the issuer's own tranche is also monotonic in the asset's return. This assumption, commonly adopted in the finance and insurance literatures, serves as a safeguard against issuers' potential manipulation of asset cash flows — either through external lending or inter-project transfers — with the intent of diminishing payouts to investors.

For simplicity, suppose that the issuer can take two actions $a \in \{0, 1\}$, where $a = 1$ corresponds to exerting effort and where $a = 0$ corresponds to shirking. For example,

[24] There were nearly no losses on AAA-rated securities issued before 2003, but cumulative losses rose to nearly 5% for securities issued in the years 2006-2008.

the founder of a start-up might work hard to ensure its success, or she might not work given that most of the returns were promised to investors. In another example, a bank needs incentives to carefully screen the loan-takers behind a pool of mortgages.[25]

Clearly, the incentives to exert effort in such cases crucially depend on the precise structure of the issued securities: based on this intuition, the Dodd-Frank Act of 2010 requires issuers to retain an equity share of no less than 5% of the aggregate credit risk of assets they securitize.

We denote the cost associated with each action by κ_a and without loss of generality we assume that $0 = \kappa_0 < \kappa_1$. The distribution of project's return depends on the issuer's effort and is denoted by H_a, with density $h_a = H_a'$. We assume that the action $a = 1$ leads to a higher return than the action $a = 0$ in the sense of the hazard rate stochastic order.[26] This assumption implies that H_1 and H_0 are ordered in (strict) first-order stochastic dominance.

Theorem 4 below shows that the optimal solution to the problem with moral hazard takes a simple form: For each possible action, consider the securities that are optimal without moral hazard. We show that, if the high action is incentive compatible for the seller given the retained profit under its associated distribution of returns, then the same securities as without moral hazard remain optimal. Otherwise the low action is optimal combined with its associated menu of securities that were optimal without moral hazard.

For a given pair of securities $R_l, R_h : \mathbb{R}_+ \to \mathbb{R}_+$ purchased by low and high risk-tolerance investors, the expected surplus retained by the issuer equals

$$\int_X [1 - (f_h R_h'(x) + f_l R_l'(x))](1 - H_a(x))dx\,.$$

Denote by $\bar{R}(x) = x - (f_h R_h(x) + f_l R_l(x))$ the share retained by the issuer if the asset's return equals x. It is optimal for the issuer to exert high effort if and only if the retained profit increases more than the cost when taking the high action:

$$\int_X \bar{R}'(x)(H_0(x) - H_1(x))dx \geq \kappa_1\,.$$

We first analyze an auxiliary problem. Let

$$\mathcal{R}_{\pi,a} = \left\{R\colon \int_X R'(x)(1 - H_a(x))dx = \pi, 0 \leq R'(x) \leq 1,\ R(0) = 0\right\}$$

[25] Keys et al.[2010] empirically argue that borrowers whose mortgage loan was more likely to be securitized were less thoroughly screened before the subprime crisis.

[26] Two random variables with distributions H_0, H_1 are ordered in the hazard rate order if $\frac{1-H_0(x)}{h_0(x)} < \frac{1-H_1(x)}{h_1(x)}$ for all $x \in X$.

be the set of doubly monotonic, absolutely continuous securities that lead to a profit of π when taking the action $a \in \{0,1\}$ (ignoring the effort cost). Denote by $\bar{\pi} = \int x dH_1(x)$ the return of the asset under high effort. In the auxiliary problem we fix the issuer's profit at to some level $\pi \in \big(0, \bar{\pi}\big)$ given that $a = 1$ is taken, and find the retained profit function $\bar{R}$ that minimize the issuer's incentive to deviate to $a = 0$, e.g. we solve

$$\max_{\bar{R} \in \mathcal{R}_{\pi,1}} \int_X \bar{R}'(x)(H_0(x) - H_1(x))dx \,. \tag{1}$$

Finally, denote by $R^{\pi}_{equity}(x) = \max\{x - p, 0\}$ a call option security with strike price p (or equity) such that its expected value under the distribution H_1 equals π.

Proposition 4 R^{π}_{equity} *is the unique solution to the auxiliary problem* (1).

Intuitively, as a call option is the security that is most sensitive to high returns, it maximizes the issuers incentives to exert effort. We can now define $\kappa^*(\pi)$ as the highest cost κ_1 such that the issuer can be incentivized to take action $a = 1$ while being promised an expected retained profit of π

$$\kappa^*(\pi) = \max_{\bar{R} \in \mathcal{R}_{\pi,1}} \int_X \bar{R}'(x)(H_0(x) - H_1(x))dx \,.$$

As H_0 is first-order stochastically dominated by H_1, and as we have shown above that the solution to the above problem is a call option, it follows that κ^* is a strictly increasing function: more profit retained by the issuer leads to better incentives to exert effort. We are now able to state our main result of this section: the moral hazard problem has *no* influence on the basic structure of the optimal securities when the issuer is risk-neutral.

Theorem 4 *Let* (R^a_l, R^a_h) *denote the optimal menu of senior-junior debt issued for the distribution of returns* H_a*, in the design problem without moral hazard. Let* $\bar{R}^a(x) = x - f_l R^a_l - f_h R^a_h$ *denote the equity part kept by the issuer, and let*

$$\kappa^{\dagger} = \int_X (\bar{R}^1)'(H_0(x) - H_1(x))dx.$$

(i) If $\kappa_1 > \kappa^{\dagger}$*, then the optimal contract specifies issuer's action* $a = 0$ *and securities* (R^0_l, R^0_h) *for investors,*

(ii) If $\kappa_1 < \kappa^{\dagger}$ *the optimal contract specifies issuer's action* $a = 1$ *and securities and securities* (R^1_l, R^1_h) *for investors for the buyers.*

The cost $\kappa^{\dagger}$ is the largest possible one such that the issuer does not want to deviate to the low action given (R^1_l, R^1_h), the optimal menu for H_1 without moral hazard.

Since moral hazard poses an additional constraint on the issuer, and since the issuer achieves the same value as without it, these are clearly also the optimal securities for any $\kappa < \kappa^\dagger$ given $a = 1$. In the Appendix we prove that deviating to low effort $a = 0$ and offering the corresponding optimal securities also reduces the issuer's profit.

If the effort cost exceeds $\kappa^\dagger$ then the securities (R_l^1, R_h^1) do not provide sufficient incentives for the issuer to take the action $a = 1$.[27] As the retained profit forms a call option, Proposition 4 implies that there is no other security that provides sufficient incentives to implement $a = 1$. Thus, the optimal contract must implement $a = 0$, and we establish in the Online Appendix that the constraint coming from the moral hazard problem is slack, implying that the optimal securities are (R_l^0, R_h^0). In either case, the optimal securities are the same as they would have been in the absence of the moral hazard problem if the action of the issuer was exogenously fixed.

Remark: The above "irrelevance" of the moral hazard problem relies on the fact that, in the benchmark problem without moral hazard, it is already efficient for the issuer to retain equity. By Proposition 4 this is also the optimal way of providing incentives to exert effort. If the issuer is more risk-averse than some investors, then he would not retain equity in the absence of a moral hazard problem, and its introduction may change the design of the optimally issued securities - we leave for further resarch this interesting topic and the examination of the related regulatory scope and impact.

6.3 Security Design by a Risk-Averse Issuer

Consider now the possibility that the issuer is also risk-averse, and her risk preference is represented by a convex distortion function g. In this subsection, we also restrict attention to doubly monotonic securities. The analysis utilizes a counterpart to Theorem 1 (see Gershkov et al. [2023]) showing that equity is the most variable security among double monotonic securities with a given expected return.[28] Let $\mathcal{R}_\gamma^D$ denote the set of all feasible, doubly monotonic securities with an expected payment γ, and let x^o solve $\mathbb{E}[(x - x^o)_+] = \gamma$. Note that $R_{equity}^{\gamma,D}(x) = (x - x^o)_+$ represents an *equity* contract included in $\mathcal{R}_\gamma^D$ and let $R_{debt}^{\gamma,D}(x)$ denote a debt contract in the same set.

Lemma 2 (Gershkov et al. [2023], Theorem 2) *Consider any security $R \in \mathcal{R}_\gamma^D$. Then $R_{equity,}^{\gamma,D}(x) \preceq R(x) \preceq R_{debt}^{\gamma,D}(x)$, where $\preceq$ denotes second-order stochastic dominance.*

[27]This can be seen as the case where the Dodd-Frank regulation has a bite.

[28]If only monotonicity is imposed, the "live-or-die" contract, as studied by Innes [1990], emerges as the least preferred asset by any risk-averse agent. This contract, defined as $R(x) = x\mathbf{1}_{x\geq k}$, transfers ownership of the asset only if its return exceeds a threshold k. It does not satisfy double monotonicity and differs fundamentally from equity.

We distinguish two cases (1) where the issuer is more risk-averse than the agreesive investors; and (2) the case where the issuer is more risk-averse than the conservative investors:

Case 1: The issuer is less risk-averse than the conservative investors, i.e., g is less convex than g_l. If the issuer is also less risk-averse than the aggressive investors, i.e., if g is less convex than g_h, then our previous analysis for a risk-neutral issuer applies. Novel findings arise for the case where the issuer is more risk-averse than the aggressive investors, i.e., if g is more convex than g_h. Thus, we focus below on the case where the seller's risk aversion is intermediate between those of the two types of investors.

Proposition 5 *Suppose that $c > f_h$ and that g is more convex than g_h. Let $\tilde{x}_l$, $\tilde{x}_h$ denote the solutions to*

$$\begin{aligned}
\int_0^{\tilde{x}_l} g_l(1 - H(x))dx &= c - f_h \\
\frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1 - H(x))dx &= \frac{1}{f_l}\int_0^{\tilde{x}_l} g_h(1 - H(x))dx + \frac{1-c}{f_l}
\end{aligned}$$

respectively. The optimal menu is given by $\tilde{t}_l = \frac{c-f_h}{f_l}$, $\tilde{t}_h = 1$, and

$$\tilde{R}_l(x) = \begin{cases} \frac{x}{f_l} & \textit{for } x \leq \tilde{x}_l \\ \frac{\tilde{x}_l}{f_l} & \textit{otherwise} \end{cases},$$

$$\tilde{R}_h(x) = \begin{cases} 0 & \textit{for } x \leq \tilde{x}_h \\ \frac{x-\tilde{x}_h}{f_h} & \textit{otherwise} \end{cases}.$$

The conservative investor still receives the safest part of the asset, i.e., a senior debt. However, the aggressive investor, who is now the least risk-averse party, receives the equity, which, according to Lemma 2, represents the most risky part of the asset. It follows that the part of the asset that is kept by the designer, $x - f_l R_l(x) - f_h R_h(x)$, takes now the form of junior debt, and is given by:

$$R^*(x) = \begin{cases} 0 & \text{for } x \leq \tilde{x}_l \\ x - \tilde{x}_l & \text{for } \tilde{x}_l < x \leq \tilde{x}_h \\ \tilde{x}_h - \tilde{x}_l & \text{otherwise} \end{cases}.$$

Case 2: If the issuer is more risk-averse than the conservative investors, the maximization problem is slightly different but can still be solved in an analogous manner. Here, we focus on the case where the gain from trade is sufficiently large, ensuring that the issuer always seeks to extract as much money from investors as possible.

If the project is sufficiently profitable, the issuer can extract the full amount from the investors, i.e., $t_l = t_h = 1$. In this case, the issuer's problem reduces to minimizing

$$\int_X \left[f_l R_l'(x) + f_h R_h'(x)\right] g(1 - H(x))\, dx,$$

subject to the same set of constraints as in Case 1, but with different values of t_h and t_l. Following an analogous analysis to Case 1, it can be verified that the designer will offer equity to the aggressive investors, junior debt to the conservative investors, and retain the senior debt. The corresponding cutoff points are jointly determined by the binding constraints (IR-l) and (IC-h), as in Case 1.

If collecting the full amount proves infeasible and given the assumption of sufficient gain from trade, the issuer will aim to sell the entire asset, i.e., $f_l R_l'(x) + f_h R_h'(x) = 1$, in order to maximize her revenue. Using similar arguments as those in Lemma 3, one can show that the optimal menu still features $t_h = 1$. The problem is then reduced to maximizing the amount t_l collected from the low type, subject to all relevant constraints.

This problem is essentially the same as the baseline model case where only one type is needed to finance the project (see Section 4.1, Proposition 1). The solution involves offering a debt contract to the conservative type and allocating the remaining asset (in the form of equity) to the aggressive type. The issuer, being the most risk-averse, sells all of the assets to the investors and retains only cash. A more detailed analysis of this problem is provided in the Supplemental Appendix.

6.4 Private Budgets

We consider here investors who have two possible types of budgets: $b = \beta < 1$ with probability p and $b = 1$ with probability $1 - p$. The individual budgets are privately known, and, for each agent, the perceived distribution of his budget is independent of the distribution of the agent's risk type. Under this assumption, the analysis for more budget types remains essentially the same. For later use, we also define the average budget as $\bar{\beta} = p\beta + 1 - p$. To ensure that the project can be financed, we assume $\bar{\beta} > c$.

Let $x_{l,1}^*$ denote the solution to the participation constraint of the conservative investor, who has a budget of 1:

$$\int_0^{x_{l,1}^*} g_l(1 - H(x))\, dx = c - f_h\bar{\beta}.$$

Let $x_{h,1}^*$ denote the solution to the IC constraint of the aggressive investor, who has a budget of 1, under the assumption that he can only pretend to be a conservative investor with the same budget type:

$$\frac{1}{f_h\bar{\beta}}\int_{x^*_{l,1}}^{x^*_{h,1}} g_h(1-H(x))\,dx = \frac{1}{f_l\bar{\beta}}\int_0^{x^*_{l,1}} g_h(1-H(x))\,dx + \frac{\bar{\beta}-c}{f_l\bar{\beta}}.$$

Theorem 5 *If $\bar{\beta} \geq c > f_h\bar{\beta}$, the menu of securities described below is optimal:*

$$R^*_{l,1}(x) = \begin{cases} \frac{x}{f_l\beta} \text{ for } x \leq x^*_{l,1} \\ \frac{x^*_{l,1}}{f_l\beta}, \text{ otherwise} \end{cases}$$

$$R^*_{h,1}(x) = \begin{cases} 0, \text{ for } x \leq x^*_{h,1} \\ \frac{x-x^*_{l,1}}{\beta f_h}, \text{ for } x^*_{l,1} \leq x \leq x^*_{h,1} \\ \frac{x^*_{h,1}-x^*_{l,1}}{\beta f_h}, \text{ otherwise} \end{cases}$$

$$R^*_{l,\beta}(x) = \beta R^*_{l,1}(x) \text{ and } R^*_{h,\beta}(x) = \beta R^*_{h,1}(x) \text{ for all } x.$$

Moreover, the price of securities are given by $t_{l,1} = \frac{c-f_h\bar{\beta}}{f_l\bar{\beta}}$, $t_{h,1} = 1$, $t_{l,\beta} = \beta t_{l,1}$, $t_{h,\beta} = \beta t_{h,1}$.

It is still optimal for the issuer to offer senior debt to the conservative investor with a high budget of 1, and junior debt to the aggressive type with a high budget 1. However, the issuer now introduces an additional option for the more budget-constrained investors: these individuals can purchase a share $\beta < 1$ of the corresponding debt at a share β of the price charged to high-budget investors.

The details of the proof can be found in the Supplemental Appendix. We provide here an outline: We first derive the optimal mechanism for the scenario in which agents have heterogeneous yet publicly known budgets (while risk preferences remain private to the agents). Next, we demonstrate that this mechanism remains implementable even when the budget is private information for each agent, Thus, it must be optimal in this context as well. This result underscores the notion that, in a large market, small investors do not derive any informational advantage/rent by keeping their budget information private.

6.5 More than Two Types

Suppose now that there are N different kinds of investors. Each one is characterized by a type $\theta_n \in \Theta = \{\theta_1, \theta_2, ..., \theta_N\}$ that determines the distortion function g_n. Each type occurs with a probability $f_n > 0$, such that $\sum_{n=1}^{N} f_n = 1$. We further assume that the investors' risk attitudes are ordered: for each $n > 1$, g_{n-1} is a convex transformation of g_n. We consider the case where all N types are needed to finance the project, i.e., $1 - f_1 < c$. This is without loss of generality: if only $m < N$ types are needed, the analysis is the same as the one for the m least risk averse types.[29]

We examine direct mechanisms $(R_n, t_n)_{n=1}^N$. By using similar arguments as that for Lemma 3, one can verify that in the optimal mechanism, $t_n = 1$ for all $n \geq 2$ and $t_1 = \frac{c-1+f_1}{f_1}$. The optimal mechanism has a waterfall structure with several "mezzanine" tranches:

Theorem 6 *Suppose that $c > 1 - f_1$ and let x_1^* denote the solution to:*

$$\int_0^{x_1^*} g_1(1 - H(x))dx = c - 1 + f_1,$$

x_2^* denote the solution to:

$$\frac{1}{f_2}\int_{x_1^*}^{x_2^*} g_2(1 - H(x))\, dx = \frac{1}{f_1}\int_0^{x_1^*} g_2(1 - H(x))\, dx + \frac{1-c}{f_1}$$

and x_n^* ,$n \geq 3$, denote the solution to

$$\frac{1}{f_n}\int_{x_{n-1}^*}^{x_n^*} g_n(1 - H(x))dx = \frac{1}{f_{n-1}}\int_{x_{n-2}^*}^{x_{n-1}^*} g_n(1 - H(x))dx$$

respectively. The optimal menu of securities is given by $t_1^* = \frac{c-1+f_1}{f_1}$, $t_n^* = 1$ for any $n \geq 2$, and by

$$R_1^*(x) = \begin{cases} \frac{x}{f_1}, & \text{for } x \leq x_1^* \\ \frac{x_1^*}{f_1}, & \text{otherwise} \end{cases}$$

$$R_n^*(x) = \begin{cases} 0, & \text{for } x \leq x_{n-1}^* \\ \frac{x - x_{n-1}^*}{f_n}, & \text{for } x_{n-1}^* \leq x \leq x_n^* \\ \frac{x_n^* - x_{n-1}^*}{f_n}, & \text{otherwise} \end{cases} \text{ for any } n \geq 2.$$

Local incentive compatibility of the above structure follows from the same arguments as in the two-type case. Global incentive compatibility is ensured because, by assumption, for any $i < j$, g_i is more convex than g_j, which implies that an investor with type θ_i is always willing to pay more for a risk reduction than one with type θ_j. Moreover, the structure of the issued securities establishes a complete ordering of asset riskiness: the most risk-averse agent receives the safest portion of the asset (the most senior debt), the second most risk-averse agent receives the safest portion of the remaining asset, and so on for each subsequent type. Together, these two conditions ensure that if a type θ_{i+1} has no incentive to deviate to purchase R_i^*, the same holds for any higher investor

[29] This observation implies that larger issues (higher c) consist of more tranches.

type. Thus, local incentive compatibility implies global incentive compatibility. The detailed proof is provided in the Supplemental Appendix.

7 Conclusion

We have analyzed a novel security design model where an issuer raises capital from a population of heterogeneous, risk-averse, and budget-constrained investors. The issuer sells securities backed by an underlying asset with stochastic returns. Investors assess risk according to the important class of spectral risk measures, including expected shortfall. Investors differ in their risk appetites and in their budgets, both of which are their private information. All agents (issuers and investors) possess the same information about the distribution of the asset's returns.

In this environment, we used the tools of mechanism design in order to derive the optimal security design. We found that the optimal mechanism partitions the asset's realized cash flow into several securities conforming to the commonly observed practice of tranching, where senior claims are fully paid before subordinate ones. We explicitly derived the structure of the optimal menu of offered securities such as senior debt, junior debt and equity, and described how it depends on the model's main features such as the asset's volatility, the financing need, and the investors' heterogenous degrees of risk aversion.

Appendix

Proof of Lemma 1. Suppose that there exists an optimal menu $(R_\theta(x), t_\theta)_{\theta\in\Theta}$ for which $f_h t_h + f_l t_l > c$ and $t_l > 0$. This implies that $R_l(x) \neq 0$ on a set of strictly positive measure. Then, we can construct another menu $(\tilde{R}_\theta(x), \tilde{t}_\theta)_{\theta\in\Theta}$ such that $\tilde{R}_h(x) = R_h(x)$ for all x and $\tilde{R}_l(x) = (1-\varepsilon)R_l(x)$ for all x and for some $\varepsilon > 0$.

Let $\tilde{t}_h = t_h$, and

$$\tilde{t}_l = t_l - \varepsilon \int_X R_l'(x) g_l (1 - H(x)) dx$$

There clearly exists a sufficiently small ε such that $f_h\tilde{t}_h + f_l\tilde{t}_l \geq c$. It is then easy to verify that the newly constructed mechanism is implementable as long as the original mechanism is implementable. Moreover, as

$$f_l(t_l - \tilde{t}_l) = f_l\varepsilon \int_X R_l'(x) g_l (1 - H(x)) dx < f_l\varepsilon \int_X R_l'(x)(1 - H(x)) dx,$$

we can conclude that the new mechanism is strictly more profitable than the original one. Thus, the original mechanism could not have been optimal, yielding a contradiction. The case where $t_l = 0$ can be proved in a similar way - we omit here the details.
■

Proof of Proposition 1. Let

$$V(R_h) = \int_X R_h'(x) g_h (1 - H(x)) dx$$

denote an aggressive investor's utility from holding security R_h, and let

$$C(R_h) = \int_X R_h'(x)(1 - H(x)) dx$$

denote the cost to the issuer of providing such a security (in terms of foregone cash-flow from the asset). Suppose that the debt contract R_h^* defined in the statement of the Proposition 1 is not optimal. Then, there exists another feasible mechanism $(\tilde{R}_h, t_h)$ such that $V(\tilde{R}_h) = V(R_h^*) = \frac{c}{f_h}$ so that (IR-h) binds, and such that $C(\tilde{R}_h) < C(R_h^*)$. It is clear that $\tilde{R}_h$ cannot be another debt contract. Then, by Theorem 1 there exists a debt contract R_h^D with cutoff x^{**} such that R_h^D second-order stochastically dominates any security R_h having the same provision cost $C(R_h) = C(\tilde{R}_h)$. Since the investor is risk-averse, we obtain that

$$V(R_h^D) \geq V(\tilde{R}_h) = V(R_h^*) = \frac{c}{f_h}.$$

where the equality follow from the construction of $\tilde{R}_h$). The above inequality, together with the observation that both R_h^* and R_h^D are debt contracts, imply that the interest rate offered in R_h^D must be higher than the one offered by R_h^*, so that $x^{**} \geq x^*$. This also implies that the cost of provision is higher $C(R_h^D) \geq C(R_h^*) > C(\tilde{R}_h)$, yielding a contradiction to the construction of $\tilde{R}_h$. ■

Proof of Lemma 3. Suppose that there exists a solution to the relaxed problem such that $t_l > 0$ and $t_h < 1$. Then, we can construct another menu $(\tilde{R}_\theta(x), \tilde{t}_\theta)_{\theta\in\Theta}$ that transfers a share ε, $0 < \varepsilon < 1$, of the asset designed for conservative investors to the asset designed to aggressive investors, i.e.

$$\tilde{R}_h(x) = R_h(x) + \frac{\varepsilon f_l}{f_h} R_l(x) \ , \ \tilde{R}_l(x) = (1-\varepsilon) R_l(x)$$

for all x. Further, let

$$\tilde{t}_h = t_h + \frac{\varepsilon f_l}{f_h} \int_X R_l'(x) g_h(1 - H(x)) dx \ , \ \tilde{t}_l = t_l - \varepsilon \int_X R_l'(x) g_l(1 - H(x)) dx.$$

For sufficiently small ε, $\tilde{t}_h < 1$ and $\tilde{t}_l > 0$. It is easy to verify that the new mechanism is implementable as long as the original mechanism is. Moreover, the change in the seller's profit is given by:

$$(f_h t_h + f_l t_l) - (f_h \tilde{t}_h + f_l \tilde{t}_l) = f_l \varepsilon \int_X R_l'(x) \left[g_h(1 - H(x)) - g_l(1 - H(x) \right] dx > 0.$$

Therefore, the original mechanism cannot be the solution to (Problem Q), yielding a contradiction. ■

Proof of Theorem 2

In order to prove Theorem 2, we first derive the optimal mechanism for the relaxed problem where we do **not** impose the incentive constraint for a low type (IC-l), **nor** the participation constraint for a high type, (IR-h). We later check that the obtained solution to the relaxed problem indeed satisfies these omitted constraints. Formally, the relaxed problem is:

$$
\begin{aligned}
\text{(Problem Q)} \quad & \min_{(R_\theta,t_\theta)_{\theta\in\Theta}} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)](1-H(x))dx \right\} \\
\text{s.t.} \; & \int_X R_l'(x) g_l (1-H(x))dx = t_l \\
& \int_X \left[R_h'(x) - R_l'(x)\right] g_h (1-H(x))dx = t_h - t_l \\
& f_l t_l + f_h t_h = c \\
& R_h'(x), R_l'(x) \geq 0, \; \forall x \in X \\
& f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x, \; \forall x \in X \\
& R_h(0) = R_l(0) = 0
\end{aligned}
$$

The first equality is the (IR-l) constraint, the second one is (IC-h), the third line is (FC') and the last two lines ensure feasibility. The following Lemma demonstrates that, in any solution to the relaxed problem, all aggressive, high types invest their whole budget, i.e. $t_h = 1$.

Lemma 3 *Suppose that $(R_\theta, t_\theta)_{\theta\in\Theta}$ is a solution to (Problem Q). If $t_l > 0$, then $t_h = 1$.*

It follows from the above lemma that, in the optimal mechanism, the seller raises f_h (in total) from the aggressive investors and still needs to raise $f_l t_l = c - f_h$ from the conservative investors. That is, $t_l = \frac{c-f_h}{f_l}$. Given this insight, the relaxed problem Q can be further simplified into:

$$
\begin{aligned}
\text{(Problem } Q') \quad & \min_{R_l,R_h} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)](1-H(x))dx \right\} \\
\text{s.t.} \; & \int_X R_l'(x) g_l (1-H(x))dx = \frac{c-f_h}{f_l} \\
& \int_X [R_h'(x) - R_l'(x)] g_h (1-H(x))dx = t_h - t_l = \frac{1-c}{f_l} \\
& R_h'(x), R_l'(x) \geq 0, \; \forall x \in X \\
& f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x, \; \forall x \in X \\
& R_h(0), R_l(0) = 0,
\end{aligned}
$$

Below, we outline the solution to (Problem Q'), which also serves as a solution to the designer's original problem.

Proposition 6 *Let x_l^*, x_h^* denote the solutions to:*

$$\int_0^{x_l^*} g_l(1 - H(x))dx = c - f_h$$

and

$$\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1 - H(x))dx = \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H(x))dx + \frac{1-c}{f_l}$$

respectively. The solution to (Problem Q') is given by:

$$R_l^*(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \le x_l^* \\ \frac{x_l^*}{f_l}, \text{ otherwise} \end{cases}$$

$$R_h^*(x) = \begin{cases} 0, \text{ for } x \le x_l^* \\ \frac{x - x_l^*}{f_h}, \text{ for } x_l^* \le x \le x_h^* \\ \frac{x_h^* - x_l^*}{f_h}, \text{ otherwise} \end{cases}$$

Proof of Proposition 6. For notational convenience, let

$$\phi(x) = f_l R_l'(x) + f_h R_h'(x).$$

denote the *slope* of the offered aggregate securities, and observe that

$$\phi(x) - R_l'(x) = f_h\left[R_h'(x) - R_l'(x)\right].$$

Intuitively, $\phi(x)$ is the share of an additional dollar of the project's return that is allocated to investors if the current return equals x. Then (Problem Q') can be rewritten as:

$$\begin{aligned}
&(\text{Problem } Q'') \min_{\phi, R_l}\left\{\int_X \phi(x)(1 - H(x))dx\right\} \\
&s.t. \int_X R_l'(x) g_l(1 - H(x))dx = \frac{c - f_h}{f_l} \\
&\int_X [\phi(x) - R_l'(x)] g_h(1 - H(x))dx = \frac{(1-c)f_h}{f_l} \\
&\phi(x) \ge f_l R_l'(x) \ge 0, \ \forall x \in X \\
&\int_0^x \phi(z)dz \le x, \ \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

We further relax the above problem by ignoring the third constraint, i.e. that says that the share given to both investor types must exceed the share given to conservative investors. The problem then becomes:

$$\begin{aligned}
&(\text{Problem } Q''') \min_{\phi, R_l} \left\{ \int_X \phi(x)(1-H(x))dx \right\} \\
&\quad s.t. \int_X R_l'(x) g_l (1-H(x))dx = \frac{c-f_h}{f_l} \\
&\quad \int_X [\phi(x) - R_l'(x)] g_h (1-H(x))dx = \frac{(1-c)f_h}{f_l} \\
&\quad \int_0^x \phi(z)dz \leq x, \ \forall x \in X \\
&\quad \phi(x), R_l'(x) \geq 0, \ \forall x \in X \\
&\quad R_l(0) = 0
\end{aligned}$$

If the solution to the above problem satisfies all the constraints of Problem Q'' then it is also a solution to Problem Q''. For solving Problem Q''' we proceed in two steps:

Step 1: We first keep the function R_l' fixed. Then we need to solve:

$$\begin{aligned}
&\min_{\phi} \left\{ \int_X \phi(x)(1-H(x))dx \right\} \\
&s.t. \int_X \phi(x) g_h (1-H(x))dx = \int_X R_l'(x) g_h (1-H(x))dx + \frac{(1-c)f_h}{f_l} \\
&\int_0^x \phi(z)dz \leq x, \ \forall x \in X \\
&\phi(x) \geq 0, \forall x \in X
\end{aligned}$$

Since g_h is convex, the same argument as in the proof for Proposition 1 yields that the optimal average security corresponds to a debt contract, and is given by $\phi(x) = \mathbf{1}_{x \leq x_h^*}$ where x_h^* solves

$$\int_0^{x_h^*} g_h (1-H(x))dx = \int_X R_l'(x) g_h (1-H(x))dx + \frac{(1-c)f_h}{f_l}.$$

Step 2: Next, the seller must optimally chooses the function R_l in order to minimize x_h^* (i.e., relax as much as possible the isoperimetric constraint) while satisfying all the other remaining constraints. Minimizing x_h^* is equivalent to the minimization problem

$$\min_{R_l} \left\{ \int_X R_l'(x) g_h (1-H(x))dx + \frac{(1-c)f_h}{f_l} \right\}$$

under the same constraints. Since $\frac{(1-c)f_h}{f_l}$ is a constant, the issuer's problem in this second step reduces to:

$$\begin{aligned}
&\min_{R_l} \left\{ \int_X R_l'(x) g_h(1 - H(x)) dx \right\} \\
s.t. &\int_X R_l'(x) g_l(1 - H(x)) dx = \frac{c - f_h}{f_l} \\
&f_l \int_0^x R_l'(z) dz \leq x \\
&R_l'(x) \geq 0, \ \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

Recall that g_l is a convex transformation of g_h. By assumption, there exists an increasing and convex function k such that $g_l(z) = k(g_h(z))$. It must be the case that $k(0) = 0$ and $k(1) = 1$.

Consider a new, artificial asset whose return is governed by the distribution $\tilde{H} : X \to [0, 1]$ defined by

$$1 - \tilde{H}(x) = g_h(1 - H(x)) \text{ for all } x \in X$$

The above problem can be then rewritten as:

$$\begin{aligned}
&\min_{R_l} \left\{ \int_X R_l'(x)(1 - \tilde{H}(x)) dx \right\} \\
s.t. &\int_X R_l'(x) k(1 - \tilde{H}(x)) dx = \frac{c - f_h}{f_l} \\
&f_l \int_0^x R_l'(z) dz \leq x, \ \forall x \in X \\
&R_l'(x) \geq 0, \ \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

We can then apply the same argument as in Step 1, and obtain that the solution to the above problem is $(R_l^*)'(x) = \frac{1}{f_l} \mathbf{1}_{x \leq x_l^*}$ where x_l^* solves

$$\frac{1}{f_l} \int_0^{x_l^*} g_l(1 - H(x)) dx = \frac{c - f_h}{f_l}.$$

It directly follows that

$$\phi^*(x) = \frac{1}{f_l} \mathbf{1}_{x \leq x_h^*}$$

where x_h^* solves

$$\begin{aligned}\int_0^{x_h^*} g_h(1-H(x))dx =& \int_X (R_l^*)'(x) g_h(1-H(x))dx + \frac{(1-c)f_h}{f_l} \\ =& \frac{1}{f_l}\int_0^{x_l^*} g_h(1-H(x))dx + \frac{(1-c)f_h}{f_l},\end{aligned} \tag{2}$$

which is equivalent to

$$\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1-H(x))dx = \frac{1}{f_l}\int_0^{x_l^*} g_h(1-H(x))dx + \frac{1-c}{f_l}$$

as desired. In addition, we obtain.

$$(R_h^*)'(x) = \frac{1}{f_h}(\phi(x) - f_l R_l'(x)) = \frac{1}{f_h}\mathbf{1}_{x_l^* \le x \le x_h^*}.$$

To conclude, we have obtained that the optimal menu of securities for (Problem Q''') is given by:

$$R_l^*(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \le x_l^* \\ \frac{x_l^*}{f_l}, \text{ otherwise} \end{cases}$$

$$R_h^*(x) = \begin{cases} 0, \text{ for } x \le x_l^* \\ \frac{x - x_l^*}{f_h}, \text{ for } x_l^* \le x \le x_h^* \\ \frac{x_h^* - x_l^*}{f_h}, \text{ otherwise} \end{cases}$$

In order to verify that the above menu is also a solution to (Problem Q'), we still need to check that the ignored constraint,

$$\phi(x) \ge f_l R_l'(x) \ge 0$$

also holds. For the last inequality to hold, it suffices to verify that $x_l^* \le x_h^*$, and this follows from Equation (2):

$$\begin{aligned}\int_0^{x_h^*} g_h(1-H(x))dx =& \frac{1}{f_l}\int_0^{x_l^*} g_h(1-H(x))dx + \frac{(1-c)f_h}{f_l} \\ \ge& \int_0^{x_l^*} g_h(1-H(x))dx.\end{aligned}$$

■

Proof of Theorem 2. In order to prove that R_l^* and R_h^* as described in Proposition 6 are the optimal securities, we still need to show that the omitted constraints, namely (IR-h) and (IC-l), are also satisfied. The fact that (IR-h) holds follows directly from

(IR-l) and (IC-h). (IC-l) requires that

$$\int_X [R_h'(x) - R_l'(x)]\, g_l(1 - H(x))dx \leq t_h - t_l = \frac{1-c}{f_l}$$
$$\Leftrightarrow \int_X [R_h'(x) - R_l'(x)]\,[g_h(1 - H(x)) - g_l(1 - H(x))]dx \geq 0$$

The equivalence holds because

$$\int_X [R_h'(x) - R_l'(x)]\, g_h(1 - H(x))dx = \frac{1-c}{f_l}$$

by the (IC-h) constraint. In order to prove (IC-l), we proceed as follows:

For any fixed $y \in X$, consider two distributions defined on the interval $[0, y]$ by

$$\pi_{yh}(x) = \frac{\int_0^x g_h(1 - H(z))dz}{\int_0^y g_h(1 - H(z))dz}; \quad \pi_{yl}(x) = \frac{\int_0^x g_l(1 - H(z))dz}{\int_0^y g_l(1 - H(z))dz}$$

The ratio of the respective densities is given by

$$\frac{\pi_{yh}'(x)}{\pi_{yl}'(x)} = \frac{\int_0^y g_l(1 - H(z))dz}{\int_0^y g_h(1 - H(z))dz} \cdot \frac{g_h(1 - H(x))}{g_l(1 - H(x))}$$

This ratio is increasing in x because we assumed that g_l is a convex transformation of g_h, which implies that $\frac{g_h(x)}{g_l(x)}$ is decreasing. This further implies that $\pi_{yh}(x) \succeq_{LR} \pi_{yl}(x)$ where LR denotes the likelihood ratio stochastic order. It is well-known (see Shaked and Shanthikumar [2007], Theorems 1.B.1, page 18 and 1.C.1, page 43) that the likelihood ratio stochastic order implies the hazard rate order, and that the latter implies the usual first order stochastic dominance. Hence we obtain that $\pi_{yh}(x) \succeq_{FOSD} \pi_{yl}(x)$ for each $y \in X$.

Note that

$$R_h'(x) - R_l'(x) = \frac{1}{f_h}\mathbf{1}_{x_l^* \leq x \leq x_h^*} - \frac{1}{f_l}\mathbf{1}_{x \leq x_l^*} = \begin{cases} -\frac{1}{f_l},\ x \leq x_l^* \\ \frac{1}{f_h},\ x_l^* \leq x \leq x_h^* \\ 0,\ x \geq x_h^* \end{cases}$$

is an increasing function on $[0, x_h^*]$. Applying the above observation about stochastic dominance to $y = x_h^*$, and recalling that the expectation of an increasing function increases under a FOSD shift, we obtain that:

$$\frac{\int_0^{x_h^*}[R_h'(x) - R_l'(x)]g_h(1 - H(x))dx}{\int_0^{x_h^*} g_h(1 - H(z))dz} \geq \frac{\int_0^{x_h^*}[R_h(x) - R_l'(x)]g_l(1 - H(x))dx}{\int_0^{x_h^*} g_l(1 - H(z))dz}$$

As $g_h(1-H(x)) \geq g_l(1-H(x))$ for all x we have

$$\frac{1}{\int_0^{x_h^*} g_h(1-H(z))dz} \leq \frac{1}{\int_0^{x_h^*} g_l(1-H(z))dz}$$

Together with $R_h'(x) - R_l'(x) = 0$ for $x \geq x_h^*$, the two inequalities above together imply that

$$\int_X \left[R_h'(x) - R_l'(x)\right] g_h(1-H(x))dx \geq \int_X \left[R_h'(x) - R_l'(x)\right] g_l(1-H(x))dx$$

as desired. ■

In order to prove Proposition 2, we first need a Lemma. Consider two assets, x and y, with distributions of returns H_x and H_y, respectively, such that either x FOSD y or x SOSD y.

Lemma 4 *Let $\hat{x}$ denote the solution to $\int_0^{\hat{x}}(1-H_x(z))dz = \int_0^{y^*}(1-H_y(z))dz$ - Then $\int_0^{\hat{x}} g_h(1-H_x(z))dz \geq \int_0^{y^*} g_h(1-H_y(z))dz$.*

Proof of Lemma 4. Define

$$H_x^D(t) = \begin{cases} H_x(t) \text{ for } x \leq \hat{x} \\ 1, \text{ otherwise} \end{cases} \quad \text{and } H_y^D(t) = \begin{cases} H_y(t) \text{ for } t \leq y^* \\ 1, \text{ otherwise} \end{cases}$$

H_x^D (H_y^D) describes the distribution of a debt contract that is backed by asset $x(y)$ with cutoff $\hat{x}$ (y^*). The expected values of these two debt contracts are, by construction, the same:

$$\begin{aligned}\int_0^{\bar{x}}(1-H_x^D(z))dz &= \int_0^{\hat{x}}(1-H_x^D(z))dz + \int_{\hat{x}}^{\bar{x}}(1-H_x^D(z))dz \\ &= \int_0^{\hat{x}}(1-H_x(z))dz + \int_{\hat{x}}^{\bar{x}}(1-1)dz = \int_0^{y^*}(1-H_y(z))dz = \int_0^{\bar{x}}(1-H_y^D(z))dz\end{aligned}$$

By the assumption that asset x SOSD asset y, and by the definition of $\hat{x}$, we know that $\hat{x} \leq y^*$. Further, for any $s \in (\hat{x}, y^*)$ it holds that:

$$\int_0^s(1-H_x^D(z))dz = \int_0^{\hat{x}}(1-H_x^D(z))dz = \int_0^{y^*}(1-H_y^D(z))dz > \int_0^s(1-H_y^D(z))dz$$

For any $s < \hat{x}$ it holds that:

$$\int_0^s(1-H_x^D(z))dz = \int_0^s(1-H_x(z))dz \geq \int_0^s(1-H_y(z))dz = \int_0^s(1-H_y^D(z))dz$$

which directly follow from the assumption that x SOSD y.

We can then conclude that H_x^D SOSD H_y^D. As investors are risk averse, we obtain

$$\int_0^{\hat{x}} g_h(1 - H_x(z))dz = \int_0^{\hat{x}} g_h(1 - H_x^D(z))dz \geq \int_0^{y^*} g_h(1 - H_y^D(z))dz.$$

as desired. ■

Proof of Proposition 2. We give the proof for SOSD. The proof for FOSD is similar and we omit the details.

Consider two assets, x and y, with distributions of returns H_x and H_y, respectively, such that either x FOSD y or x SOSD y. Letting x_l^*, y_l^*, x_h^* and y_h^* denote the solutions to:

$$\text{(IR-l)} \int_0^{x_l^*} g_l(1 - H_x(t))dt = \int_0^{y_l^*} g_l(1 - H_y(t))dt = c - f_h$$

and

$$\begin{aligned}\text{(IC-h)} \ &\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1 - H_x(t))dt - \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H_x(t))dt \\ =&\frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1 - H_x(t))dt - \frac{1}{f_l}\int_0^{y_l^*} g_h(1 - H_x(t))dt = 1 - \frac{c - f_h}{f_l}\end{aligned}$$

Then, $x_l^* \leq y_l^*$ directly follows from the definition of SOSD and from the agents' risk aversion. The rest of proof consists of 3 steps.

Step 1: Let $\tilde{x}$ denote the solution to

$$\int_0^{\tilde{x}} [1 - H_x(t)]dt = \int_0^{y_h^*} [1 - H_y(t)]dt$$

It follows from Lemma 4 that:

$$\int_0^{\tilde{x}} g_h(1 - H_x(t))dt \geq \int_0^{y_h^*} g_h(1 - H_y(t))dt$$

Step 2: We next show that

$$\int_0^{x_l^*} g_h(1 - H_x(t))dt \leq \int_0^{y_l^*} g_h(1 - H_y(t))dt$$

This follows from

$$\int_0^{x_l^*} g_l(1 - H_x(t))dt = \int_0^{y_l^*} g_l(1 - H_y(t))dt$$

and from the assumption that g_l is more convex than g_h (i.e., because type l, the conservative type, is more risk-averse than type h, the aggressive type) .

Step 3: Steps 1 and 2 imply together that

$$\int_{x_l^*}^{\tilde{x}} g_h(1 - H_x(t))dt \geq \int_{y_l^*}^{y_h^*} g_h(1 - H_y(t))dt$$

which further implies

$$\begin{aligned}
&\frac{1}{f_h}\int_{x_l^*}^{\tilde{x}} g_h(1 - H_x(t))dt - \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H_x(t))dt \\
&\geq \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1 - H_y(t))dt - \frac{1}{f_l}\int_0^{y_l^*} g_h(1 - H_y(t))dt
\end{aligned}$$

Recall that x_h^* and y_h^* solve

$$\begin{aligned}
&\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1 - H_x(t))dt - \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H_x(t))dt \\
&= \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1 - H_y(t))dt - \frac{1}{f_l}\int_0^{y_l^*} g_h(1 - H_y(t))dt
\end{aligned}$$

It follows that $x_h^* \leq \tilde{x}$, and thus that

$$\int_0^{x_h^*}(1 - H_x(z))dz \leq \int_0^{\tilde{x}}(1 - H_x(z))dz = \int_0^{y_h^*}(1 - H_y(z))dz$$

which proves Part 2 of Proposition 2. It also follows that

$$\begin{aligned}
\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1 - H_x(z))dz &= \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H_x(z))dz + (1 - \frac{c - f_h}{f_l}) \\
&\leq \frac{1}{f_l}\int_0^{y_l^*} g_h(1 - H_y(z))dz + (1 - \frac{c - f_h}{f_l}) = \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1 - H_y(z))dz,
\end{aligned}$$

which proves Part 3 of Proposition 2.

The only remaining task is to show that under a FOSD risk $x_h^* - x_l^* \leq y_h^* - y_l^*$. Suppose not: then

$$x_h^* - x_l^* > y_h^* - y_l^* \equiv \Delta$$

Since $x_l^* < y_l^*$ and g_h is non-decreasing, we obtain:

$$\begin{aligned}
\int_{x_l^*}^{x_h^*} g_h(1 - H_x(z))dz &> \int_{x_l^*}^{x_l^*+\Delta} g_h(1 - H_x(z))dz \\
&\geq \int_{y_l^*}^{y_l^*+\Delta} g_h(1 - H_x(z))dz \geq \int_{y_l^*}^{y_h^*} g_h(1 - H_y(z))dz
\end{aligned}$$

where the last inequality follows from FOSD. This leads to a contradiction since we proved above that

$$\int_{x_l^*}^{x_h^*} g_h(1 - H_x(z))dz \geq \int_{y_l^*}^{y_h^*} g_h(1 - H_y(z))dz.$$

We therefore conclude that $x_h^* - x_l^* \leq y_h^* - y_l^*$ if x FOSD y. ■

Proof of Proposition 3. Take any c, let $x_l^*(c)$ denote the cutoff point of the senior debt, which solves (IR-l):

$$\int_0^{x_l(c)} g_l(1 - H(z))dz = c - f_h\,.$$

Taking the derivative with respect to c twice in the above expression yields:

$$g_l(1 - H(x_l(c))) \cdot x_l'(c) = 1$$
$$-g_l'(1 - H(x_l(c))h(x_l(c)) \cdot [x_l'(c)]^2 + g_l(1 - H(x_l(c))) \cdot x_l''(c) = 0$$

The second equation yields $x_l''(c) \geq 0$. Moreover, clearly $x_l(0) = 0$. Hence the corresponding interest rate, $\frac{x_l(c)}{c-f_h} - 1$ is increasing in the financing need c.

Similarly, $x_h(c)$, which denotes the cutoff point for the junior debt, is given by (IC-h):

$$\frac{1}{f_h}\int_{x_l(c)}^{x_h(c)} g_h(1 - H(t))dt - \frac{1}{f_l}\int_0^{x_l(c)} g_h(1 - H(t))dt = 1 - \frac{c - f_h}{f_l}$$
$$\Rightarrow \frac{1}{f_h}\int_{x_l(c)}^{x_h(c)} g_h(1 - H(t))dt = 1 + \frac{1}{f_l}\int_0^{x_l(c)} [g_h(1 - H(t)) - g_l(1 - H(t))]dt$$

As c increase, $x_l(c)$ increases. Moreover, $g_h(1 - H(t)) - g_l(1 - H(t)) \geq 0$. As the right hand side of the equation increases, the left hand side of the equation, $\int_{x_l(c)}^{x_h(c)} g_h(1 - H_x(z))dz$ must increase as well.

Finally, we want to prove that $x_h(c) - x_l(c)$ also increases as c increases. Suppose that this is not the case. Then there must exist $c_1 > c_2$ such that $x_h(c_1) - x_l(c_1) < x_h(c_2) - x_l(c_2)$. Let $\Delta = x_h(c_2) - x_l(c_2)$. Since $g_h(1 - H(t))$ is decreasing in t and because $x_l(c_1) > x_l(c_2)$, we have

$$\int_{x_l(c_1)}^{x_h(c_1)} g_h(1 - H_x(z))dz < \int_{x_l(c_1)}^{x_l(c_1)+\Delta} g_h(1 - H_x(z))dz$$
$$\leq \int_{x_l(c_2)}^{x_l(c_2)+\Delta} g_h(1 - H_x(z))dz = \int_{x_l(c_2)}^{x_h(c_2)} g_h(1 - H_x(z))dz$$

which contradicts the result obtained above. Therefore, it must hold that $x_h(c) - x_l(c)$ increases as c increases. ■

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Supplemental Appendix:
Optimal Security Design for Risk-Averse Investors

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Supplemental Appendix A: Reinvestment is Never Strictly Optimal

As noted in Section 3.1, one can extend the model by allowing the issuer to use capital collected from investors in order to increase some payments to other investors. Then, the feasibility constraint becomes:

$$\sum_{\theta \in \{l,h\}} f_\theta \, R_\theta(x) \leq x + \sum_\theta f_\theta t_\theta - c.$$

In order to demonstrate that such a strategy is never strictly optimal, we consider two cases.

Case 1: Suppose that only aggressive investors participate in the optimal mechanism and that the issuer raises $c + \Delta$, where $\Delta > 0$. Then, the optimal issued security must be a solution to the following problem:

$$\begin{aligned}
&\min_{R_h} \left\{ f_h \int_X R_h'(x)(1 - H(x))dx \right\} \\
\text{s.t.} &\int_X R_h'(x) g_h (1 - H(x))dx = \frac{c + \Delta}{f_h} \\
&R_h'(x) \geq 0, && \forall x \in X \\
&f_h \int_0^x R_h'(z)dz \leq x + \Delta, && \forall x \in X \\
&R_h(0) = 0
\end{aligned}$$

Using the same arguments as that in the proof for Proposition 1, we can deduce that the solution to the above problem is a debt contract given by:

$$R_h(x) = \begin{cases} \frac{x+\Delta}{f_h} & \text{if } x \leq x^* \\ \frac{x^*+\Delta}{f_h} & \text{otherwise} \end{cases}$$

where x^* is the solution to $\int_0^{x^*} g_h(1 - H(z))\, dz = c$.

The above security is equivalent to a direct reimbursement of Δ coupled with the same debt contract as described in Proposition 1.

Case 2: Suppose the issuer raises $c+\Delta$ to fund a project whose return is governed by the distribution $H(x)$, and both investor types participate in the optimal mechanism. Analogous to Case 1, this can be reinterpreted as an alternative scenario where the designer intends to raise $c+\Delta$ to fund a project whose return is given by the distribution $H(x-\Delta)$. Technically, given that Lemma 3 remains valid, we obtain that $t_h = 1$ and that $t_l = \frac{c-f_h+\Delta}{f_l}$. The formulation of the designer's problem, excluding (IC-l) and (IR-h), becomes then:

$$
\begin{aligned}
\min_{(R_\theta,t_\theta)_{\theta\in\Theta}} & \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)](1-H(x))dx \right\} \\
\text{s.t.} & \int_X R_l'(x) g_l (1-H(x))dx = t_l = \frac{c-f_h+\Delta}{f_l} \\
& \int_X \left[R_h'(x) - R_l'(x)\right] g_h (1-H(x))dx = t_h - t_l = 1 - \frac{c-f_h+\Delta}{f_l} \\
& R_h'(x), R_l'(x) \geq 0, \ \forall x \in X \\
& f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x + \Delta, \ \forall x \in X \\
& f_l R_l(0) + f_h R_h(0) \leq \Delta
\end{aligned}
$$

By using the same arguments as that in the proof for Theorem 2, we obtain that the solution to the above problem is given by

$$
R_l^*(x) = \begin{cases} \frac{x+\Delta}{f_l} \text{ for } x \leq x_l^* \\ \frac{x_l^*+\Delta}{f_l}, \text{ otherwise} \end{cases}
$$

$$
R_h^*(x) = \begin{cases} 0, \text{ for } x \leq x_l^* \\ \frac{x-x_l^*}{f_h}, \text{ for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^*-x_l^*}{f_h}, \text{ otherwise} \end{cases}
$$

In the above formulas, x_l^* and x_h^* are defined as in Theorem 2. Analogous to Case 1, the present situation can be interpreted as equivalent to directly reimbursing the low type agents with the additionally raised amount Δ, followed by issuing to both types the same securities as in Theorem 2.

By the above, we can conclude that it is without loss to solely concentrate on asset-backed securities.

Supplemental Appendix B: Proofs for Section 6: Extensions

Proofs for results in Section 6.1 (No-Purchasing Limits)

Proof of Theorem 3. We let

$$\phi(x) = f_h R_h'(x) + f_l R_l'(x)$$

be the slope of the offered aggregate securities as in the proof of Proposition 6. It follows that

$$\frac{1}{f_h}\left[\phi(x) - \frac{cf_l}{c-f_h}R_l'(x)\right] = R_h'(x) - \frac{f_l}{c-f_h}R_l'(x)$$

and the issuer's relaxed problem becomes:

$$\begin{aligned}
&\min_{R_h,R_l}\left\{c\int_X \phi(x)(1-H(x))dx\right\} \\
s.t. &\int_X R_l'(x)g_l(1-H(x))dx = 1 \\
&\int_X \phi(x)g_h(1-H(x))dx = \frac{f_l c}{c-f_h}\int_X R_l'(x)g_h(1-H(x))dx \\
&\phi(x) \geq f_l R_l'(x) \geq 0, && \forall x \in X \\
&c\int_0^x \phi(z)dz \leq x, && \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

We solve the above problem analogously to the method we used in Section 4.2. First, fixing the security R_l, the optimal average slope ϕ is given by $\phi(x) = \mathbf{1}_{x\leq \hat{x}_h}$ where $\hat{x}_h$ is the solution to

$$\int_0^{\hat{x}_h} g_h(1-H(x))dx = \frac{f_l c}{c-f_h}\int_X R_l'(x)g_h(1-H(x))dx.$$

This yields a debt contract. Next, the seller must choose the optimal security R_l in order to minimize

$$\int_X \phi(x)(1-H(x))dx$$

This is equivalent to

$$\begin{aligned}
&\min_{R_l}\left\{\int_X R_l'(x)g_h(1-H(x))dx\right\} \\
s.t. &\int_X R_l'(x)g_l(1-H(x))dx = \frac{c-f_h}{f_l} \\
&\int_0^x R_l'(z)dz \leq \frac{1}{f_l}\min\{x,\hat{x}_h\}, \forall x \in X
\end{aligned}$$

Since g_l is a convex transformation of g_h, then, again by the same argument as in Section 4.2, we obtain that the optimal security R_l satisfies $R_l'(x) = \frac{1}{f_l}\mathbf{1}_{x\leq \hat{x}_l}$ where $\hat{x}_l$ is the solution to the equation

$$\int_0^{\hat{x}_l} g_l(1-H(x))dx = c - f_h.$$

It can be then easily computed that $\hat{x}_h$ solves

$$\int_0^{\hat{x}_h} g_h(1-H(x))dx = \frac{c}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx.$$

which is equivalent to

$$\frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1-H(x))dx = \frac{1}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx$$

as desired. It follows that the optimal securities R_l and R_h are now given by:

$$R_l^*(x) = \begin{cases} \frac{x}{f_l}, \text{ for } x \leq \hat{x}_l \\ \frac{\hat{x}_l}{c-f_h}, \text{ otherwise} \end{cases}$$

and

$$R_h^*(x) = \begin{cases} 0, \text{ for } x \leq \hat{x}_l \\ \frac{x-\hat{x}_l}{f_h}, \text{ for } \hat{x}_l \leq x \leq \hat{x}_h \\ \frac{\hat{x}_h-\hat{x}_l}{f_h}, \text{ otherwise} \end{cases}$$

The omitted (IC-l) constraint holds by the same argument as in the proof for Theorem 2. ∎

Lemma 5 *The interest rate for senior debt* $\frac{\hat{x}_l}{c-f_h} - 1$ *is greater than the interest rate for junior debt* $\frac{\hat{x}_h-\hat{x}_l}{f_h} - 1$*, in the case without purchase limits.*

Proof. To prove that this is indeed the case, recall that the (IC-h) constraint now reads:

$$\int_X [R_h'(x) - R_l'(x)]\, g_h(1-H(x))dx = 0$$
$$\Leftrightarrow \frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1-H(x))dx = \frac{1}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx$$

As the function $g_h(1-H(x))$ is non-negative and decreasing, the following chain of inequalities immediately follows from (IC-h):

$$\frac{g_h(1-H(\hat{x}_l))}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} dx > \frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1-H(x))dx$$
$$= \frac{1}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx > \frac{g_h(1-H(\hat{x}_l))}{c-f_h}\int_0^{\hat{x}_l} dx$$

The above inequalities further imply that

$$\frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} dx > \frac{1}{c-f_h}\int_0^{\hat{x}_l} dx \Rightarrow \frac{\hat{x}_h-\hat{x}_l}{f_h}-1 > \frac{\hat{x}_l}{c-f_h}-1$$

as desired. ■

Proofs for results in Section 6.2 (Moral Hazard)

Proof of Proposition 4. By the Lagrangian principle there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that the optimal $\bar{R}$ maximizes

$$\int_X \bar{R}'(x)\big((1+\lambda)[1-H_1(x)]-[1-H_0(x)]\big)dx = \int_X \bar{R}'(x)\big((1+\lambda)Z(x)-1\big)[1-H_0(x)]dx\,,$$

where $Z(x) = (1-H_1(x))/(1-H_0(x))$. As the distributions H_0, H_1 are ordered in the hazard rate order, Z is increasing. Furthermore, $Z(0) = 1$ such that if $\lambda \geq 0$ we get that $\bar{R}(x) = x$ which contradicts $\pi < \int xh_1(x)dx$. If $\lambda \leq -1$ this implies that $\bar{R}(x) = 0$ which contradicts $\pi > 0$. Thus, $\lambda \in (-1, 0)$ which implies that $(1+\lambda)Z(x)-1$ changes sign at most once from negative to positive which in turn implies that $\bar{R}'$ is 0 below some level and 1 above that level which means $\bar{R}$ is a call option. As H_1 admits a density, there is a unique call option with expectation π which is thus the unique solution to (1). ■

Proof of Theorem 4. Part (ii): Suppose that $\kappa_1 < \kappa^\dagger$. In this case given the securities (R_l^0, R_h^0) it is not optimal for the seller to deviate to the action $a = 0$. Thus, when taking the action $a = 1$ the seller obtains the same profit with and without moral hazard which implies that (R_l^1, R_h^1) must be optimal securities. We are left to check that the seller can not benefit from offering the securities which are optimal under the action $a = 0$ and taking the action $a = 0$. Doing so would decrease her profit as

$$\int_X (\bar{R}^0)'(1-H_0(x)) - (\bar{R}^*)_1'(1-H_1(x))dx + \kappa_1$$
$$\leq -\left[\int_X (\bar{R}^1)'(H_0(x)-H_1(x))dx - \kappa_1\right] = -[\kappa^\dagger - \kappa_1] < 0\,.$$

Part (i): If κ_1 exceeds the above bound, then since κ^* is strictly increasing, incentivizing the action $a = 1$ would require leaving the designer with a strictly higher profit than what could be achieved with the ability to commit to an action. As this is not feasible, it becomes impossible to incentivize $a = 1$. It remains to demonstrate that $a = 0$ can be incentivized. We note that the benefit of taking action $a = 0$ over action $a = 1$ is given as

$$\begin{aligned}
&\int_X (\bar{R}^0)'(1 - H_0(x))dx - \int_X (\bar{R}^0)'(1 - H_0(x))dx + \kappa_1 \\
&\qquad = -\left[\int_X (\bar{R}^0)'(H_0(x) - H_1(x))dx - \kappa_1\right] \\
&\qquad \geq -\left[\int_X (\bar{R}^1)'(H_0(x) - H_1(x))dx - \kappa_1\right] \\
&\qquad = \kappa_1 - \kappa^\dagger > 0\,.
\end{aligned}$$

Thus, the agent does not want to deviate to taking the high action given the securities (R_l^0, R_h^0) are sold to investors. ■

Proofs for Section 6.3 Security Design by a Risk-Averse Issuer

Proof of Proposition 5. Restricting attention to doubly monotonic contracts, and following essentially the same steps as above, the issuer's problem becomes

$$\begin{aligned}
&\min_{R_l, R_h} \left\{\int_X [f_l R_l'(x) + f_h R_h'(x)] g(1 - H(x))dx\right\} \\
&\text{s.t.} \int_X R_l'(x) g_l(1 - H(x))dx = \frac{c - f_h}{f_l} \\
&\qquad \int_X [R_h'(x) - R_l'(x)] g_h(1 - H(x))dx = t_h - t_l = \frac{1 - c}{f_l} \\
&\qquad R_h'(x), R_l'(x) \geq 0 \ \forall x \in X \\
&\qquad f_l R_l'(x) + f_h R_h'(x) \leq 1 \ \forall x \in X \\
&\qquad R_h(0), R_l(0) = 0
\end{aligned}$$

The third and fourth constraints represent the double monotonicity conditions. Together, the two conditions imply that the contract is feasible, and thus the feasibility constraint $f_l R_l(x) + f_h R_h(x) \leq x$ for all x is no longer needed.

By assumption, there exists an increasing and convex function $k(\cdot)$ such that $g(z) = k(g_h(z))$. It must be the case that $k(0) = 0$ and $k(1) = 1$.

As in the benchmark model, we first derive the optimal mechanism for the relaxed problem where we impose neither (IC-l) nor (IR-h). We later check that the obtained solution for the relaxed problem indeed satisfies these omitted constraints. Formally, the relaxed problem is:

$$\begin{aligned}
&\min_{R_l,R_h}\left\{\int_X [f_l R_l'(x)+f_h R_h'(x)]g(1-H(x))dx\right\}\\
&\text{s.t.} \int_X R_l'(x)g_l(1-H(x))dx = \frac{c-f_h}{f_l}\\
&\int_X [R_h'(x)-R_l'(x)]g_h(1-H(x))dx = t_h - t_l = \frac{1-c}{f_l}\\
&R_h'(x), R_l'(x) \geq 0 \ \forall x \in X\\
&f_l R_l'(x) + f_h R_h'(x) \leq 1 \ \forall x \in X\\
&R_h(0), R_l(0) = 0
\end{aligned}$$

The proof follows a similar procedure to that of Proposition 6. We first fix R_l, and look at the following relaxed problem:

$$\begin{aligned}
&\min_{R_h}\left\{\int_X R_h'(x)g(1-H(x))dx\right\}\\
&s.t. \int_X R_h'(x)g_h(1-H(x))dx = \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l}\\
&0 \leq R_h'(x) \leq \frac{1}{f_h}, \ \forall x \in X\\
&R_h(0) = 0
\end{aligned}$$

Consider a new, artificial asset whose return is governed by the distribution $\tilde{H} : X \to [0,1]$ defined by

$$1-\tilde{H}(x) = g_h(1-H(x)) \text{ for all } x \in X$$

Then, the above problem can be rewritten as follows:

$$\begin{aligned}
&\min_{R_h}\left\{\int_X R_h'(x)k(1-\tilde{H}(x))dx\right\}\\
&s.t. \int_X R_h'(x)(1-\tilde{H}(x))dx = \int_X R_l'(x)(1-\tilde{H}(x))dx + \frac{1-c}{f_l}\\
&0 \leq R_h'(x) \leq \frac{1}{f_h}, \ \forall x \in X\\
&R_h(0) = 0
\end{aligned}$$

Let

$$\tilde{V}(R_h) = \int_X R_h'(x)k(1-\tilde{H}(x))dx; \ \tilde{C}(R_h) = \int_X R_h'(x)(1-\tilde{H}(x))dx$$

denote the utility derived from holding security R_h by an agent whose dual risk preference is described by the distortion k, and the cost to a risk-neutral seller of issuing such a security, respectively.

The issuer's problem is thus equivalent to the design of a doubly monotonic security that **minimizes** the agent's utility while keeping the expected cost fixed. Then, by Theorem 1, the optimal security has the form of an equity:

$$\tilde{R}_h(x) = \begin{cases} 0 \text{ for } x \leq \tilde{x}_h \\ \frac{x-\tilde{x}_h}{f_h}, \text{ otherwise} \end{cases}$$

where $\tilde{x}_h$ is the solution to

$$\frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1-H(x))dx = \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l}$$

By following essentially the same procedure as in the proof of Proposition 6, we obtain that the optimal R_l takes the form of senior debt, and is given by:

$$\tilde{R}_l(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \leq \tilde{x}_l \\ \frac{\tilde{x}_l}{f_l}, \text{ otherwise} \end{cases}$$

where $\tilde{x}_l$ solves

$$\int_0^{\tilde{x}_l} g_l(1-H(x))dx = c - f_h$$

It follows that

$$\begin{aligned} \frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1-H(x))dx &= \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l} \\ &= \frac{1}{f_l}\int_0^{\tilde{x}_l} g_h(1-H(x))dx + \frac{1-c}{f_l} \end{aligned}$$

The last step is to check the menu described in Proposition 5 satisfies the ignored constraints (IR-h) and (IC-l). Note that

$$\tilde{R}_h'(x) - \tilde{R}_l'(x) = \begin{cases} -\frac{1}{f_l}, \; x \leq \tilde{x}_l \\ 0, \; \tilde{x}_l \leq x \leq \tilde{x}_h \\ \frac{1}{f_h} \; x \geq \tilde{x}_h \end{cases}$$

increases on $[0, \bar{x}]$. Then, we can use the same arguments as that in the proof of Theorem 2 to show that the two ignored constraints are satisfied. ■

Case 2: The issuer solves then

$$\begin{aligned}
&\max_{R_l,R_h} f_l t_l + f_h \\
\text{s.t.} \quad & \int_X R_l'(x) g_l(1-H(x))\,dx = t_l, \\
& \int_X [R_h'(x) - R_l'(x)] g_h(1-H(x))\,dx = t_h - t_l = 1 - t_l, \\
& R_h'(x), R_l'(x) \geq 0 \quad \forall x \in X, \\
& f_l R_l'(x) + f_h R_h'(x) = 1 \quad \forall x \in X, \\
& R_h(0), R_l(0) = 0.
\end{aligned}$$

The constraints are respectively (IR-l), (IC-h), double monotonicity, and feasibility constraints. Note that (IC-h) can be rewritten as

$$\begin{aligned}
f_h t_l &= f_h - f_h \int_X [R_h'(x) - R_l'(x)] g_h(1-H(x))\,dx \\
&= \int_X [1 - f_h R_h'(x) + f_h R_l'(x)] g_h(1-H(x))\,dx = \int_X R_l'(x) g_h(1-H(x))\,dx.
\end{aligned}$$

so the issuer's problem can be rewritten as

$$\begin{aligned}
&\min_{R_l} \int_X R_l'(x) g_h(1-H(x))\,dx \\
\text{s.t.} \quad & 0 \leq R_l'(x) \leq 1 \quad \forall x \in X, \\
& R_h(0) = 0.
\end{aligned}$$

The problem is essentially the same as the baseline model case where only one type is needed to finance the project (Section 5.1, Proposition 1). The solution is to give a debt contract to the conservative type and the remaining asset (which takes the form of equity) to the aggressive type. The issuer, who is the most risk-averse, sells all of the assets to investors and only keeps cash.

Proofs for Section 6.4 Private Budgets

Proof of Theorem 5. The proof consists of three main steps.

Step 1: Suppose that the agents' budget types are public information while the risk types remain the agents' private information, as before. We show that there exists an optimal menu such that $R^*_{l,\beta}(x) = \beta R^*_{l,1}(x)$, $R^*_{h,\beta}(x) = \beta R^*_{h,1}(x)$ for all x, $t^*_{l,\beta} = \beta t^*_{l,1}$, and $t^*_{h,\beta} = \beta t^*_{h,1}$.

Step 1-a: We first show that if there exists an optimal mechanism $(R^*_{\theta b}, t^*_{\theta b})$ for which $R^*_{l,\beta}(x) \neq \beta R^*_{l,1}(x)$, then we can construct another optimal mechanism $(\tilde{R}^*_{\theta b}, \tilde{t}^*_{\theta b})$ such that $\tilde{R}^*_{l,\beta}(x) = \beta \tilde{R}^*_{l,1}(x)$.

Observe that in any optimal mechanism, the constraints (IR-$l1$), (IR-$l\beta$), (IC-$h1$), and (IC-$h\beta$) must all bind:

$$
\begin{aligned}
(IR-l1): & \int_X R'_{l,1}(x)g_l(1-H(x))dx = t_{l,1} \\
(IR-l\beta): & \int_X R'_{l,\beta}(x)g_l(1-H(x))dx = t_{l,\beta} \\
(IC-h1): & \int_X [R'_{h,1}(x) - R'_{l,1}(x)]g_h(1-H(x))dx = t_{h,1} - t_{l,1} \\
(IC-h\beta): & \int_X [R'_{h,\beta}(x) - R'_{l,\beta}(x)]g_h(1-H(x))dx = t_{h,\beta} - t_{l,\beta}
\end{aligned}
$$

Putting the above equations together yields:

$$
\begin{aligned}
& p\int_X [R'_{h,\beta}(x) - R'_{l,\beta}(x)]g_h(1-H(x))dx + (1-p)\int_X [R'_{h,1}(x) - R'_{l,1}(x]g_h(1-H(x))dx \\
& = pt_{h,\beta} + (1-p)t_{h,1} - \int_X [pR'_{l,\beta}(x) + (1-p)R'_{l,1}(x)]g_l(1-H(x))dx \\
\Rightarrow & \int_X [pR'_{h,\beta}(x) + (1-p)R'_{h,1}(x)]g_h(1-H(x))dx - pt_{h,\beta} - (1-p)t_{h,1} \\
& = \int_X [pR'_{l,1}(x) + (1-p)R'_{l,\beta}(x)][g_h(1-H(x)) - g_l(1-H(x)]dx
\end{aligned}
$$

Thus, as long as the total asset assigned to conservative investors remains unchanged, i.e. as long as

$$
pR^*_{l,\beta}(x) + (1-p)R^*_{l,1}(x) = p\tilde{R}^*_{l,\beta}(x) + (1-p)\tilde{R}^*_{l,1}(x)\ \forall x,
$$

we can construct another incentive compatible mechanism where the total asset assigned to the aggressive investors and their total expected payment are also unchanged. If the original mechanism was optimal, so is the new one.

Step 1-b: By (IR-$l1$) and (IR-$l\beta$), in the newly constructed mechanism $(\tilde{R}^*_{\theta b}, \tilde{t}^*_{\theta b})$, $\tilde{R}^*_{l,\beta}(x) = \beta\tilde{R}^*_{l,1}(x)$ implies $\tilde{t}^*_{l\beta} = \beta\tilde{t}^*_{l1}$.

Step 1-c: Suppose now that $\tilde{t}^*_{h,\beta} \neq \beta\tilde{t}^*_{h,1}$. This means that the budget of aggressive investors are not exhausted. By using similar arguments to those in Lemma 3, it can be verified that such a mechanism cannot be optimal.

Finally, steps (1.a)-(1.c) together imply that $\tilde{R}^*_{h,\beta}(x) = \beta\tilde{R}^*_{h,1}(x)$.

Step 2: By Step 1, assuming that the agents' budget types are public information, we can restrict attention to the class of menus that satisfy $R^*_{l,\beta}(x) = \beta R^*_{l,1}(x)$, $R^*_{h,\beta}(x) = \beta R^*_{h,1}(x)$ for all x, $t^*_{l,\beta} = \beta t^*_{l,1}$, and $t^*_{h,\beta} = \beta t^*_{h,1}$. Then by following essentially the same arguments as in the proof of Theorem 2, we can show that the mechanism described in Theorem 5 is optimal in this class.

Step 3: The remaining step is to verify that, even when budget types are private information, the mechanism described in Theorem 5 is implementable, and thus optimal. It is clear that the individual rationality constraints for all types remain the same, so that they are satisfied. Moreover, as in the public budget setting, no agent has incentive to pretend to be another agent with the same budget type but different risk type. We show below that either an agent has no incentive to pretend to be another agent with the same risk type but different budget type, or he is unable to do so:

a Type $l1$ has no incentive to pretend to be of type $l\beta$ since in either case he will earn a payoff of 0 (this follows from the homogeneity of dual utility).

b Type $l\beta$ may not have enough money ($\beta < t_{l1}$) to pretend to be type $l1$. Even if $\beta > t_{l1}$, type $l\beta$ still has no incentive to pretend to be of type $l1$ since in either case he will earn a payoff of 0.

c Type $h\beta$ cannot pretend to be type $h1$ since he does not have enough money to do so ($\beta < 1 = t_{h1}$).

d Finally, type $h1$ has no incentive to pretend to be of type $h\beta$ since:

$$\begin{aligned}\int_X R'_{h,1}(x)g_h(1-H(x))dx - t_{h,1} &= \frac{1}{\beta}[\int_X R'_{h,\beta}(x)g_h(1-H(x))dx - t_{h,\beta}] \\ &> \int_X R'_{h,\beta}(x)g_h(1-H(x))dx - t_{h,\beta}\end{aligned}$$

Finally, no type of investor wants here to misreport in both dimensions: since an agent who misreports his budget essentially “adopts” the utility function of that budget type, the observation follows from the standard incentive compatibility constraint with respect to deviations in the risk type only. To conclude, even if budget types are private information, the mechanism described in Theorem 5 is implementable, and yields the same expected profit as in the case with public budget. Therefore, it must be an optimal mechanism. ■

Proofs for the results of Section 6.5 (More than Two Types): Here each investor is characterized by a type $\theta_n \in \Theta = \theta_1, \theta_2...\theta_N$, that determines his risk preferences according to distortion function g_n. Each type occurs with a probability $f_n > 0$, such that $\sum_{n=1}^{N} f_n = 1$. We further assume that the investors' risk attitudes are ordered: for each $n > 1$, g_{n-1} is a convex transformation of g_n. We consider the case where all N types are needed to finance the project, i.e., $1 - f_1 < c$, and we examine direct mechanisms $(R_n, t_n)_{n=1}^N$. Following similar arguments as those for the benchmark model, we consider mechanisms that satisfy the following constraints:

$$\sum_{n=1}^{N} f_n t_n = c. \tag{FC}$$

For a type θ_n agent not to deviate and claim to be of type θ_k, it must hold that

$$\int_X \left[R_n'(x) - R_k'(x)\right] g_n(1 - H(x))dx \geq t_n - t_k \tag{IC-nk}$$

Similarly, in order to ensure that a type θ_k agent purchases the security offered to him instead of pursuing an outside option that is normalized here to yield zero utility, it must be the case that

$$\int_X R_n'(x) g_n(1 - H(x))dx \geq t_n \tag{IR-n}$$

The feasibility constraint requires for each $x \in X$

$$\sum_{n=1}^{N} f_n R_n(x) \leq x$$

which is equivalent to $R_n(0) = 0$ for $\theta_n \in \Theta$ and

$$\sum_{n=1}^{N} \int_0^x f_n R_n'(z)dz \leq x \tag{Feasibility}$$

for all $x > 0$. Additionally, we require that the return of the security is increasing in the return of the underlying asset: for any $\theta_n \in \Theta$

$$R_n'(x) \geq 0 \tag{M}$$

and for all x, and that each type θ_n has a limited budget of 1:

$$t_n \leq 1. \tag{BC}$$

The designer's problem is to

$$\min_{R_1,\ldots,R_N} \sum_{n=1}^{N} \int_X f_n R_n'(x)(1 - H(x))\, dx$$

subject to all the above-mentioned constraints. By using similar arguments as those for Lemma 3, one can verify that in the optimal mechanism, $t_n^* = 1$ for all $n \geq 2$ and

$t_1^* = \frac{c-1+f_1}{f_1}$.

We first solve the following relaxed problem and then show that the solution to the relaxed problem is also the solution to the original problem.

$$\text{(Problem } P'\text{)} \quad \min_{R_1,\ldots,R_N} \left\{ \sum_{n=1}^{N} \int_X f_n R_n'(x)(1 - H(x))\, dx \right\}$$

$$\text{s.t.} \quad \int_X R_1'(x) g_1 (1 - H(x))\, dx = t_1^*$$

$$\int_X [R_{n+1}'(x) - R_n'(x)] g_{n+1}(1 - H(x))\, dx = t_{n+1}^* - t_n^*, \quad \forall n$$

$$R_n'(x) \geq 0, \quad \forall n \; \forall x \in X$$

$$\sum_{n=1}^{N} f_n \int_0^x R_n'(z)\, dz \leq x, \quad \forall x \in X$$

$$R_n(0) = 0, \quad \forall n; \quad t_1^* = \frac{c - 1 + f_1}{f_1}, \quad t_n^* = 1, \quad \forall n \geq 2$$

Note that in the relaxed problem, we only consider (IR-θ_1) and local incentive compatibility constraints (IC-$n+1, n$) for all n (i.e., no type has incentive to pretend to be the type just below his type). We will later verify that the solution to the relaxed problem satisfies all ignored constraints, and therefore is also the solution to the original problem.

Proposition 7 *Suppose that $c > 1 - f_1$ and let x_1^* denote the solutions to:*

$$\int_0^{x_1^*} g_1(1 - H(x))dx = c - 1 + f_1,$$

x_2^* denote the solutions to:

$$\frac{1}{f_2} \int_{x_1^*}^{x_2^*} g_2(1 - H(x))\, dx = \frac{1}{f_1} \int_0^{x_1^*} g_2(1 - H(x))\, dx + \frac{1 - c}{f_1}$$

and x_n^* for any $n \geq 3$

$$\frac{1}{f_n} \int_{x_{n-1}^*}^{x_n^*} g_n(1 - H(x))dx = \frac{1}{f_{n-1}} \int_{x_{n-2}^*}^{x_{n-1}^*} g_n(1 - H(x))dx$$

respectively. The solution to (Problem P') is given by

$$R_1^*(x) = \begin{cases} \frac{x}{f_1}, & \textit{for } x \le x_1^* \\ \frac{x_1^*}{f_1}, & \textit{otherwise} \end{cases}$$

$$R_n^*(x) = \begin{cases} 0, & \textit{for } x \le x_{n-1}^* \\ \frac{x - x_{n-1}^*}{f_n}, & \textit{for } x_{n-1}^* \le x \le x_n^* \\ \frac{x_n^* - x_{n-1}^*}{f_n}, & \textit{otherwise} \end{cases}$$

for any $n \ge 2$*.*

Proof for Proposition ??. The proof follows a procedure very similar to that of Proposition 6, so we omit some of the details here.

For notational convenience, let

$$\phi_n(x) = \sum_{k=1}^{n} f_k R_k'(x)$$

denote the *slope* of the total securities offered to the k lowest types. Then observe that

$$f_n[R_n'(x) - R_{n-1}'(x)] = \phi_n(x) - \sum_{k=1}^{n-1} f_k R_k'(x) - f_n R_{n-1}'(x).$$

Step 1 First, we keep $R_1, R_2, \ldots, R_{N-1}$ fixed. Then, we need to solve:

$$\min_{\phi_N} \left\{ \int_X \phi_N(x)(1 - H(x))\, dx \right\}$$

subject to:

$$\int_X \phi_N(x) g_N(1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{N-1} f_k R_k'(x) + f_N R_{N-1}'(x) \right] g_N(1 - H(x))\, dx,$$

$$\int_0^x \phi_N(z)\, dz \le x, \quad \forall x \in X,$$

$$\phi_N(x) \ge 0, \quad \forall x \in X.$$

By the same argument as in the proof for Proposition 6 Step 1, one can show that the optimal ϕ_N is given by $\phi_N^*(x) = \mathbf{1}_{x \le x_N^*}$, where x_N^* solves

$$\int_0^{x_N^*} g_N(1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{N-1} f_k R_k'(x) + f_N R_{N-1}'(x) \right] g_N(1 - H(x))\, dx.$$

Step 2: Observe that minimizing x_N^* is equivalent to the minimization problem:

$$\min_{R_1 \dots R_{N-1}} \int_X \left[\sum_{k=1}^{N-1} f_k R_k'(x) + f_N R_{N-1}'(x) \right] g_N(1 - H(x))\, dx$$

under the same constraints. We now fix $R_1, R_2, \dots, R_{N-2}$ and solve this new minimization problem, which is further equivalent to minimizing

$$\frac{f_{N-1}}{f_N + f_{N-1}} \phi_{N-1}(x).$$

This is because all terms in $\phi_{N-1}(x)$ except R_{N-1}' are fixed, so they can be added or subtracted from the objective function. Then we need to solve:

$$\min_{\phi_{N-1}} \left\{ \int_X \phi_{N-1}(x) g_N(1 - H(x))\, dx \right\}$$

subject to:

$$\int_X \phi_{N-1}(x) g_{N-1}(1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{N-2} f_k R_k'(x) + f_{N-1} R_{N-2}'(x) \right] g_{N-1}(1 - H(x))\, dx + t_{N-1}^* - t_{N-2}^*,$$

$$\int_0^x \phi_{N-1}(z)\, dz \leq x, \quad \forall x \in X,$$

$$\phi_{N-1}(x) \geq 0, \quad \forall xinX.$$

By the same arguments as in Step 1, we can show that the optimal ϕ_{N-1} is given by $\phi_{N-1}^*(x) = \mathbf{1}_{x \leq x_{N-1}^*}$, where x_{N-1}^* solves

$$\int_0^{x_{N-1}^*} g_{N-1}(1-H(x))\, dx = \int_X \left[\sum_{n=1}^{N-2} f_n R_n'(x) + f_{N-1} R_{N-2}'(x) \right] g_{N-1}(1-H(x))\, dx + t_{N-1}^* - t_{N-2}^*.$$

Recall that $t_n^* = 1$ for $n \geq 2$ and $t_1^* = \frac{c-1+f_l}{f_l}$.

Step 3-Step $N-1$ *(only needed for* $N \geq 4$*)*: Repeat the above step $N-3$ times. We can show that for any $n \geq 2$, the optimal $\phi_n^*(x)$ is given by $\mathbf{1}_{x \leq x_n^*}$, where x_n^* solves

$$\int_0^{x_n^*} g_n(1 - H(x))\, dx = \int_X \left[\sum_{k=1}^{n-1} f_k R_k'(x) + f_n R_{n-1}'(x) \right] g_n(1 - H(x))\, dx + t_n^* - t_{n-1}^*.$$

Step N: Now our problem is reduced to choosing the optimal R_1 to

$$\min_{R_1}\left\{\int_X R_1'(x)g_2(1-H(x))\,dx\right\}$$

subject to:

$$\begin{aligned}
&\int_X R_1'(x)g_1(1-H(x))\,dx = \frac{c-1+f_1}{f_1},\\
&f_1\int_0^x R_1'(z)\,dz \le x,\\
&R_1'(x)\ge 0, \quad \forall x\in X,\\
&R_1(0)=0.
\end{aligned}$$

We can then apply the same argument as in the proof for Proposition 6, Step 2, and obtain that the solution to the above problem is $(R_1^*)'(x)=\frac{1}{f_1}\mathbf{1}_{x\le x_1^*}$, where x_1^* solves

$$\frac{1}{f_1}\int_0^{x_1^*} g_1(1-H(x))\,dx = \frac{c-1+f_1}{f_1}.$$

In order to verify that the securities described in Proposition 6 are the solution to (Problem P'), we still need to confirm that x_n^* is increasing in n to ensure that the feasibility constraint is satisfied. This follows from, for any $n\ge 2$,

$$\begin{aligned}
\int_0^{x_n^*} g_n(1-H(x))\,dx &= \int_X\left[\sum_{k=1}^{n-1} f_kR_k'(x)+f_nR_{n-1}'(x)\right]g_n(1-H(x))\,dx + t_n^*-t_{n-1}^*\\
&=\int_0^{x_{n-1}^*} g_n(1-H(x))\,dx+\int_X f_nR_{n-1}'(x)g_n(1-H(x))\,dx+t_n^*-t_{n-1}^*\\
&\ge\int_0^{x_{n-1}^*} g_n(1-H(x))\,dx.
\end{aligned}$$

The inequality follows as $t_n^*\ge t_{n-1}^*$ and $R_{n-1}'(x)\ge 0$ for all $x\in X$. This completes the proof. ■

Proof for Theorem 6. In order to prove that the mechanism described in Theorem 6 are the optimal securities, we still need to show that the omitted constraints are also satisfied.

Step 1: We show that if (IC-$n+1$, n) holds for any n, then (IC-n, k) holds for any $n>k$.

First, observe that:

$$\int_X [R'_n(x) - R'_k(x)]\, g_n(1 - H(x))\, dx = \int_X \left[R'_n(x) - R'_{n-1}(x)\right] g_n(1 - H(x))\, dx$$
$$+ \int_X \left[R'_{n-1}(x) - R'_{n-2}(x)\right] g_n(1 - H(x))\, dx$$
$$+ \dots$$
$$+ \int_X \left[R'_{k+1}(x) - R'_k(x)\right] g_n(1 - H(x))\, dx.$$

Next, take any $i > j$. Since $R'_i - R'_j$ is increasing on $[x^*_{j-1}, x^*_i]$ and g_j is more convex than g_i, we can use the same arguments as in the proof of Theorem 2 to show that:

$$\int_X \left[R'_i(x) - R'_j(x)\right] g_i(1 - H(x))\, dx \geq \int_X \left[R'_i(x) - R'_j(x)\right] g_j(1 - H(x))\, dx.$$

Combining these results, we obtain:

$$\int_X [R'_n(x) - R'_k(x)]\, g_n(1 - H(x))\, dx \geq \int_X \left[R'_n(x) - R'_{n-1}(x)\right] g_n(1 - H(x))\, dx$$
$$+ \int_X \left[R'_{n-1}(x) - R'_{n-2}(x)\right] g_{n-1}(1 - H(x))\, dx$$
$$+ \dots$$
$$+ \int_X \left[R'_{k+1}(x) - R'_k(x)\right] g_{k+1}(1 - H(x))\, dx$$
$$\geq (t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \dots + (t_{k+1} - t_k) = t_n - t_k.$$

Thus, (IC-n, k) holds as desired.

Step 2: Take any $n > k$. We show that if (IC-n, k) holds , then (IC-k, n) also holds. The proof is essentially the same as that for Theorem 2, so we omit the details.

Step 3: The fact that (IR-k) holds follows directly from (IR-1) and (IC-n, k).

Steps 1-3 together show that all omitted constraints are satisfied. Therefore, we can conclude that the solution to (Problem P') is also the solution to the original problem. That is, the securities described in Theorem 6 are optimal. ■

Supplemental Appendix C: Risk Preferences in Financial Markets

In the main paper we assumed that investors have preferences that correspond to a class of non-expected utility models. There is ample laboratory evidence showing that expected utility does not perform well in explaining agents' risk taking behavior (Bruhin, Fehr-Duda and Epper [?], Diecidue, Wakker and Zeelenberg [?] among others). Some of these papers argue that probability distortions play an important role (see the survey of Starmer [?]), and find more support for models of rank dependent preferences than for EU preferences (Weber and Kirsner [?]). Other papers question the validity of rank-dependent utility. For example, Bernheim and Sprengler [?] argue that subjects do not adjust the weights they assign to different outcomes when the ranking of the outcomes changes, as predicted by rank-dependent theory. These authors suggest that the original Prospect Theory (PT) may fit the data in experiments better than both rank-dependent and EU preferences. Yet, as it is well known, PT violates FOSD even in very simple lotteries (see Wakker [?]). Diecidue, Wakker and Zeelenberg [?] found evidence for the presence of rank dependent preferences and elicited individual weight functions. However, they also found that the probability weights change even if the ranks do not, violating rank-dependent theory. Wu [?] illustrated a violation of the ordinal independence axiom that is a necessary property of any rank-dependent preferences[30]. Oprea [?] argues that probability weighting in lotteries stems from the complexity of lottery evaluation rather than from the attitude towards risk.

Field evidence from financial markets also provides mixed support for expected utility. Barberis, Huang and Thaler [?] showed that a combination of first-order risk aversion and narrow framing can explain the stock market participation puzzle. Probability weighting can explain several financial phenomena, such as low average returns on IPO securities (Barberis and Huang [?]). Polkovnichenko [?] showed that rank-dependent preferences are consistent with observed patterns of investment in both well-diversified and poorly-diversified portfolios of stocks. Such patterns are inconsistent with any theory, such as expected utility, in which risk attitudes stem from the curvature of the utility function only. Polkovnichenko and Zhao [?] estimated probability weighting functions from option prices assuming rank-dependent utility and cumulative prospect theory preferences. While they show that agents apply probability weighting (and hence do not follow EU), they find evidence for the inverse-S shape weighting postulated in Prospect Theory.

[30]Ordinal independence states that if two lotteries share the same upper tale, then the preference between these two lotteries remains the same even if the upper tail is substituted with another common tail.

Several studies emphasized the presence of heterogeneity of risk attitudes among decision makers. Considering both expected utility and prospect theory, von Gaudecker, van Soest, and Wengström [**?**] found that risk preferences are heterogeneous and that most of this heterogeneity cannot be explained by observables such as age, gender and education. The importance of heterogeneity in risk preferences and probability weighting was illustrated in Andrikogiannopoulou and Papakonstantinou [**?**] in the context of sport betting.

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