Optimal Security Design for Risk-Averse Investors

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Abstract

We use the tools of mechanism design, combined with the theory of risk measures, to analyze a model where a cash constrained owner of an asset with stochastic returns raises capital from a population of investors that differ in their risk aversion and budget constraints. The distribution of the asset's cash flow is assumed here to be common-knowledge: no agent has private information about it. The issuer partitions and sells the asset's cash flow into several asset-backed securities, one for each type of investor. The optimal partition conforms to the commonly observed practice of *tranching* (e.g., senior debt, junior debt and equity) where senior claims are paid before the subordinate ones. The holders of more senior/junior tranches are determined by the relative risk appetites of the different types of investors and of the issuer, with the more risk-averse agents holding the more senior tranches. Tranching endogenously arises here in an optimal mechanism because of simple economic forces: the differences in risk appetites among agents, and in the budget constraints they face.

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1 Introduction

The importance of asset-backed securities within the financial industry can hardly be overestimated: Even after the financial crisis, securitization remains a multi-trillion-dollar business. The underlying asset in securitization is typically a pool of financial obligations, such as mortgages or loans, but it can also be a cash-flow-generating fixed asset, such as a ship, aircraft, or an entire business. The profile of expected cash flows from the underlying asset is synthetically partitioned and sold into multiple *tranches*. These tranches, all backed by the same pool of assets, exhibit different risk, yield, duration, and other characteristics. The defining feature of observed tranching in practice is that each additional dollar of cash flow is allocated to a unique type of tranche in decreasing order of subordination: payments to investors conform to a *waterfall* structure where more senior claims are fully paid before junior ones start to be served.

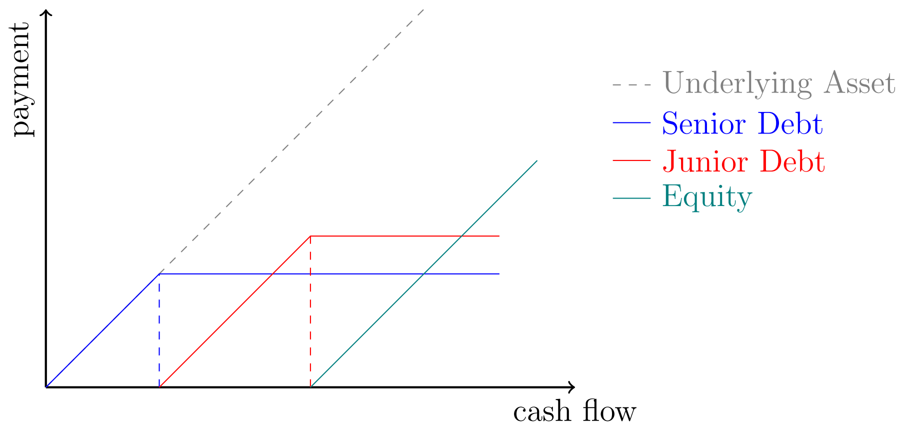

Figure 1: Tranching: Gray: Underlying Asset; Blue: Senior Debt; Red: Junior Debt; Green: Equity

Securitization is employed to fund projects with uncertain returns or to enhance capital capacity. For instance, banks are required by regulators to maintain capital according to the size and type of their loans. These “tied” reserves increase the institution’s ability to absorb potential losses but reduce its opportunities to use that capital for purposes that may generate higher returns. By securitizing assets (thereby removing them from their balance sheet), banks can decrease the reserves they must hold, thus reallocating the freed capital. Securitization by tranching caters to both conservative and more aggressive (i.e., less risk-averse) investors since it provides a variety of product choices tailored to specific investor needs in terms of duration, risks, cash-flow patterns, and yields.

In a frictionless, complete market where it is possible to trade securities with payoffs that are contingent on any conceivable event, the nature of issued securities should be, in fact, irrelevant. Since the frictionless market model is not realistic, the field of security design aims to explain optimal financial structures given prevalent market frictions. A standard theoretical argument that aims to explain tranching is that it enables a decomposition of the asset's cash flow into a component that is largely independent of a seller's private information (debt), and an information sensitive component with cash flows that are dependent on the seller's information (equity). Thus, the main friction responsible for tranching is taken to be a lemons problem, namely, the adverse selection arising from the potential informational advantage of issuer relative to investors. This kind of theory predicts which security - debt or equity - is retained by the issuer, but does not predict how the various sold securities are split among heterogenous investors. The rise and growing significance of publicly available credit ratings for securities, combined with regulated information disclosure in sales prospectuses, mitigate the aforementioned lemons problem.

The area of security design constitutes an obvious field of applications for the general theory of mechanism design. Nevertheless, the full force of this theory has been seldom applied to such settings, most probably because the objects upon which security design operates - synthetic financial instruments whose payoffs are contingent on the realizations of assets with stochastic returns - constitute relatively complex high dimensional allocation space, and because the classical theory often assumes risk-neutral agents.

In the present paper, we use the tools of mechanism design, combined with the theory of risk measures, to analyze a model where an issuer with insufficient financial resources raises capital from a population of different classes of risk-averse and budget-constrained investors by securitizing an underlying asset that has a stochastic return. We show that, in the optimal mechanism, the issuer partitions and sells the underlying asset's realized cash flow into several tranches, one for each type of risk-averse investor, such that more conservative investors are offered less risky securities.

In order to clearly differentiate our explanation from the one based on the lemons problem, the distribution of the project's cash flow is assumed here to be common-knowledge: no agent has private information about it. Thus, our model explains the emergence of the tranching technique without appealing to an asymmetry of information about the quality of the underlying asset. Tranching endogenously arises here in an optimal mechanism because of simpler and more basic economic forces: the differences in risk appetites among agents, and in the budget constraints they face. Indeed, a principal practical motivation for securitization is to appeal to investor groups with heterogeneous preferences. Roughly speaking, senior securities are designed to appeal to "constrained" or *conservative* investors who can only purchase investment-grade products, and are thus more risk-averse. These investors are bound by either transparent regulations (for example, banks, pension funds, and insurance companies are often restricted in the types of assets they may hold), or by less transparent constraints such as internal by-law restrictions/investment mandates or other portfolio/time-specific hedging requirements. Less constrained, *aggressive* investors - such as hedge funds, private equity funds and sovereign funds - are less risk averse, and are thus willing to purchase riskier securities. The junior securities that offer higher returns in exchange for more risk are designed to appeal to them. Since the reasons for being constrained in the above sense are diverse and also change over time, the same financial institution may belong to either group in a specific transaction, and the issuer may not know what is the current risk appetite of an individual investor, nor what is his present investment budget. Even if the issuer possesses such information, regulatory constraints may prevent her from using it.

Another area where our model can be applied is crowdfunding. Since May 2016, the Securities and Exchange Commission has allowed firms to issue debt (peer-to-peer lending) and equity securities. This provides unsophisticated investors a chance to participate in the securities markets, and gives small businesses an opportunity to raise funds. The Crowdfund Act includes monetary limitations for both issuers and investors: issuers may not raise more than $1,000,000 annually via crowdfunding; for investors, the maximum annual aggregate amount of crowdfunded securities that any one investor may purchase is limited, and based on a scale tied to the investor's income. Again, heterogeneous risk appetites that may not be observable to the issuer play a major role in the design of crowdfunding schemes.[1]

The above considerations imply that successful security design - that raises the needed cash for the least possible amount of foregone returns from the asset - requires the issuer to respect an incentive constraint: each investor type needs to purchase the security that is intended for its corresponding risk appetite and budget. In line with this motivation, investors in our model are heterogeneous with respect to their preferences over risks, and are privately informed about these and about their budgets. The issuer then uses a menu of securities to raise cash, while screening the various types of investors. The issuer's financing need cannot always be fully covered by the least risk-averse (aggressive) investors, and hence more risk-averse (conservative) investors must be attracted with less risky securities, yielding the sequential senior-junior security structure where each additional dollar of realized return is allocated to a unique class of investors, forming a "waterfall" structure. A higher financing need

[1]For example, here is how the Shojin Investment company advertises its services: "The key to deciding how to invest in property through crowdfunding is balancing the risks and the rewards. So, for example, if you are willing to take a slightly higher risk, you can reap significantly higher returns through Equity Crowdfunding. If your risk appetite is more conservative, you can opt for the Debt Crowdfunding option that offers the fixed return and interest."

generally leads to more tranches being offered.

Our main results explicitly derive the structure of the optimal menu of offered securities, and describe how it depends on the model's main features: the size of the financing need relative to available budgets, the relative risk aversions of all involved agents (issuer and investors), the relative frequency of investors with different risk appetites and their budget constraints. If the issuer is less risk averse than all types of investors, then the offered securities are debt contracts with different seniority and risk-return profiles. The issuer retains then an equity tranche. In contrast, if the issuer's risk aversion is such that there are potential investors who are both more or less risk averse than the issuer, then the aggressive, least risk-averse investors buy the equity tranche.

An important consequence of our main result is that, unless all assets in a pool are comonotonic (i.e., unless all assets are "bets on the same horse"), it is beneficial for an issuer to issue securities backed by the entire pooled asset rather than by its separate parts. If done optimally, such a pooled issue leads to a strictly lower financing cost.

Assuming the issuer is the least risk-averse party, we also find that if the issuer owns a stochastically better/safer asset (in terms of either a first-order or second-order stochastic shift) or if the issuer needs to raise less capital, then the optimally offered rate for senior debt decreases, and the issuer is better-off. Interestingly, in that case, the aggressive investors *always* get worse off! This is somewhat surprising since these investors are risk-averse and yet they prefer to invest in a security that is backed by a worse/more risky asset. This phenomenon is due to the effects of screening: aggressive investors earn information rents because they value the risky asset more than conservative investors. When the asset becomes less risky/better, the difference in valuations between the different types of investors decreases, and so does the information rent. This observation has interesting policy implications: market regulators often ban very risky assets in order to protect investors, but our results show that such restrictions may actually hurt risk tolerant investors.

If investors become more conservative, we show that the issued securities become riskier so that the probability of not being able to serve the outstanding debt contracts increases.

A main methodological departure from the classical finance literature - that assumes agents who are expected utility maximizers - is obtained here by endowing the agents with risk-averse, non-expected utility preferences that mirror the important class of *spectral* or *coherent distortion* risk measures (see Acerbi [2002] and Wang [1996]) that are derived via Choquet integrals of the underlying decumulative distribution (see Schmeidler [1989]). This class is obtained by taking weighted sums of *average values at risk* or averages of *expected shortfalls*, and it forms the building block for the entire set of law-invariant, coherent risk measures axiomatized by Artzner et al. [1999].[2] Thus, the risk preferences we use are closely related to recent recommendations made by the Basel Committee on Banking Supervision [2019].[3]

There is a one-to-one correspondence between spectral risk measures and the so-called *dual* risk-averse utility functionals that are better known in decision theory (see Yaari [1987]).[4] Yaari's dual utility functional form, whose formulation we use below, belongs to the class of *rank-dependent* utility functionals (see Quiggin [1982]). It uses a non-linear function to distort probabilities rather than payoffs, and weights each payoff by a weight that is decreasing in the size of the payoff.[5] Among other desirable properties, it disentangles attitudes towards risk from the marginal utility of money, that is constant.[6] An important consequence that is very appealing in our context is *additive comonotonicity*: the dual utility of holding two comonotonic securities (these are two securities whose payoff depends monotonically on the same underlying asset so that there is no hedging benefit from holding the bundle) is simply equal to the sum of the dual utilities of holding each security separately.

Another main feature that distinguishes dual utility from expected utility is *first-order risk aversion*: in the limit where the stakes become small, the risk premium vanishes linearly in the size of the risk. This is in stark contrast to any EU preference represented by a twice differentiable utility function that exhibits *second-order risk aversion*: in the small stakes limit, EU agents become risk neutral and the risk premium they demand vanishes quadratically in the size of the risk.[7] This difference can have far-reaching implications for behavior.[8] In particular, even if risks are divided into very small parts, the investors do not approach risk-neutrality as fast as in the

[2]The class of law invariant, coherent risk measures is obtained by minimizing over several possible distortions. See Safra and Segal [1998] and Kusuoka [2001].

[3]A coherent risk measure is a function that satisfies monotonicity, sub-additivity, homogeneity, and translational invariance. See Rüschendorf [2013] for the meaning of these properties when applied to risk measures.

[4]Dual utility's axiomatization replaces the classical von Neumann-Morgenstern independence axiom behind the expected utility (EU) with another axiom about mixtures of comonotonic random variables. For a good exposition on risk measures and their connections to axiomatic, non-expected utility see Arztner et al [1999] , Föllmer and Schied [2016], Chapter 4 ,or Rüschendorf [2013], Chapter 7.

[5]Guriev [2001] offers a "micro-foundation" for dual utility: a risk neutral agent who faces a bid-ask spread in the credit market will behave as if he were dual risk averse. The same happens if gains are taxed but losses are not.

[6]In the special case where the project to be financed can either succeed or fail, our results are more general and can be applied to the entire class of non-expected utility displaying Constant Risk Aversion (CRA) (see Safra and Segal [1998]) with a convex risk-premium function.

[7]See Segal and Spivak [1990] for definitions and a discussion of the various orders of risk aversion.

[8]For example, Epstein and Zin [1990] argue that first order risk aversion can resolve the *equity premium puzzle*: faced with small-stakes lotteries, a dual risk-averse (EU risk-averse) agent requires a risk premium proportional to the standard deviation (variance) of the lottery. Since the standard deviation for small risks is considerably larger than the variance it generates a higher equity premium.

standard EU model.

The rest of the paper is organized as follows: In the remainder of this section, we survey the relevant literature. In Section 2, we present the security design model with risk-averse and budget-constrained, heterogeneous investors. In Section 3, we describe and characterize feasible and incentive-compatible mechanisms in this setting. Section 4 recalls a fundamental result about debt and equity as extreme securities in the second-order stochastic dominance sense. This result is used in the main proofs below. Optimal security design via tranching is derived in Section 5. Section 6 presents several comparative statics results about changes in the optimal design when the underlying asset becomes stochastically better or safer, or less costly to implement. Section 7 offers several extensions to the basic model where we (separately) consider privately known investor budgets, a risk-averse issuer, and an issuer who takes into account the possibility of trading among investors after the initial issue. Section 8 concludes.

1.1 Related Literature

The financial literature on security design is very extensive. Coval et al. [2009] provide an accessible account of structured finance in the context of the 2007-2008 financial crisis. We also refer the reader to the recent, comprehensive survey of the scientific literature by Allen and Barbalau [2022], and will focus below on the aspects most pertinent to our paper.

Classical security design models based on asymmetric information assume that the issuer (insider) is relatively more informed than the investors (see for example Leland and Pyle [1977], Myers and Majluf [1984], Nachman and Noe [1994], DeMarzo and Duffie [1999], Malenko and Tsoy [2023][9]). This creates a lemons problem since the uninformed investors need to draw inferences about the assets' merit from the contracts proposed by the informed issuer. These models focus on the offered securities' sensitivity to the issuer's private information, and derive conditions under which debt dominates other securities. This is the basis of the renowned "pecking order" theory. In a model that incorporates noise traders and incomplete markets, Boot and Thakor [1993] showed that the issuer's expected revenue is enhanced by selling several financial claims that partition its total asset cash flows in two tranches, equity and debt, rather than selling a single claim. Roughly speaking, such a partition is profitable because it enables the decomposition of the cash flow into an information insensitive component and an information sensitive component that is dependent on the seller's information. De Marzo [2005] also investigates how the asymmetry of information interacts with the ex-ante incentives to pool various assets before securitization. DeMarzo, Frankel,

[9]This last paper assumes that investors operate under Knightian uncertainty.

and Jin [2021] extend the "pecking order" theory by studying an issuer who holds multiple assets and who designs multiple securities before and/or after she becomes informed. The authors find that it is optimal for the issuer to pool all her assets, and that two issuing strategies are optimal and equivalent: the issuer can either wait to become informed and issue a single debt equity, or she can first tranche the pooled asset into a set of prioritized debt securities and sell those tranches whose seniority exceeds an information-sensitive threshold.

Frank and Goyal [2003] and Fama and French [2005] criticize the "pecking order" theory with its main driving force - the lemons problem created by the informational asymmetry between issuer and investors - and observe that firms issue much more equity than predicted by that theory (where it should be only the a "last resort" security retained by the issuer). Ospina and Uhlig [2018] compare ex-ante credit ratings of a large set of mortgage-backed securities with (post financial crisis) "ideal" ratings given the observed outcomes and come to the conclusion that ex-ante ratings were relatively accurate (particularly on top AAA tranches). This suggests that the lemons problem was not that severe. They also show that there were nearly no losses on AAA-rated securities issued before 2003, but cumulative losses rose to nearly 5% for securities issued in the years 2006-2008. A possible explanation suggested by our model is that investors became increasingly risk-averse as the crisis approached.

A smaller literature reverses the nature of the information asymmetry: the outside investors rather than the issuer have superior information about the project's prospects. Axelson [2007] argues that this fits well situations where start-up companies seek to raise funding from professional investors or intermediaries, such as venture capital firms. Several papers following De Marzo et al [2005] (e.g., Che and Kim [2010]) study a model in which privately informed investors choose securities from a set ordered by steepness, rather than competing with securities designed by the seller as in Axelson's model. In all these studies, investors are risk-neutral, and the transaction occurs between the issuer and a unique winner, who is the only one to make a payment. In particular, risk aversion and tranching do not play a role in the obtained results.

A number of papers study security design problems with risk-neutral agents endowed with heterogeneous beliefs (see, for example, Garmaise [2001], Broer [2018], Ellis et al. [2022] and Ortner and Schmalz [2019]). For instance, Ortner and Schmalz [2019] assume that the issuer is more optimistic than the investors and only consider doubly-monotonic securities.[10] Their Corollary 1 offers relatively restrictive assumptions on the nature of the heterogeneous beliefs under which the optimally issued securities follow the standard "waterfall" structure. While this interesting class of models can

[10] This means that both the securities sold to investors and the share retained by the issuer are monotonic in the asset's cash flow.

explain the emergence of pooling assets and tranching them into structured securities, they are not fully consistent with an optimal mechanism design analysis: when it is common knowledge that market participants have diverse beliefs (i.e., when they "agree to disagree"), the issuer can arbitrage the differences in beliefs by organizing structured trades among the investors, hereby extracting the whole available surplus. In other words, in order to rationalize the use of standard securitization policies, one needs somewhat ad-hoc restrictions - that are not explicitly specified - on the class of feasible mechanisms. Our own model shares some technical features with the above (an alternative interpretation of our non-expected dual utility functionals is that agents have a distorted belief that overweights more adverse events, leading to non-linear probability weighting), but here investors are rational and the mechanism design analysis is relatively general. The waterfall structure in our environment follows from the combination between the investors' heterogeneous risk aversion and their budget constraints. Another difference between our paper and those including heterogeneous beliefs is our incomplete information assumption about risk attitudes (and budgets): this is the main driving force behind our new comparative static results whereby, for example, aggressive investors prefer the optimal securities backed by riskier assets.

Luo and Yang [2023] also consider a heterogeneous belief model but focus on coordination frictions. In their model, the value of the project depends on an unknown state, and on the total capital raised from investors. Miscoordination arises because, prior to investment, each investor receives a private and noisy signal about the unknown state, while not being able to precisely infer others' investment decisions. The authors find that the optimal issued securities follow a "waterfall" structure, such that agents with a lower perception of participation are offered more senior tranches.

As described above, somewhat surprisingly, risk aversion is not a standard feature in models of security design. For instance, it is absent in the classical models that explain the occurrence of debt contracts by appealing to costly verification, bankruptcy penalties or moral hazard (see for example Townsend [1979], Gale and Hellwig [1985], Diamond [1984] and Innes [1990]). This is mostly due to the high technical difficulty of such an analysis within the framework of expected utility. Allen and Gale [1989] and Malamud et al. [2010] study optimal security design for risk-averse investors from the point of view of risk sharing, but their models do not incorporate private information/incentive constraints. In particular, Allen and Gale show that, in their general equilibrium model where issuing securities is costly, neither debt nor equity are optimal securities. This is mainly because their risk-averse investors have preferences represented by smooth, expected utility functionals. Thus, their classical analysis cannot explain the emergence of the standard securities that are observed in practice.

Mechanism design analysis for risk-averse agents is indeed relatively complex and therefore the literature mostly focuses on the performance of fixed mechanisms, such as standard auctions. Revenue maximization with risk-averse buyers has only been studied within the EU framework by Maskin and Riley [1984] and by Matthews [1983]. Matthews [1983] restricts attention to constant absolute risk aversion (CARA) expected utility preferences, and finds that the optimal mechanism resembles a modified first-price auction where the seller sells partial insurance to bidders with high valuation, but charges an entry fee to bidders with low valuation. Maskin and Riley [1984] allow for more general risk-averse EU preferences and establish several important properties of an optimal auction without obtaining an explicit solution for their general case. Gershkov et al. [2022] assume that risk-averse bidders are equipped with dual utility (as in the present paper) and show that the optimal mechanism offers full insurance while distorting the allocation via an endogenous randomization. More related to the present study, Gershkov et al [2022] focus on classical monopolistic insurance with dual risk-averse agents, and derive the optimal menu of contracts for an insurer that maximizes revenue: in general these menus offer *layer* insurance where each additional dollar of potential loss is either fully retained by the insuree or fully passed to the insurer. Under some additional regularity assumptions, optimal contracts take the form of menus of different deductibles up to full insurance, or menus of full insurance up to different coverage limits.

Esö and White [2004] analyze EU risk-averse buyers who bid for a given risky asset in a standard auction. They assume that all buyers share the same, commonly known risk preference but receive different, privately known signals about the asset's expected value. They find that buyers exhibiting decreasing absolute risk aversion (DARA) are better off when bidding for a risky object relative to bidding for an object with a deterministic value. In contrast, we examine here the design of issuer-optimal securities in a framework where investors have privately known risk attitudes. The investors' utilities exhibit here constant absolute risk aversion CARA - under this assumption, Esö and White's bidders are indifferent - and yet, in our framework, some types strictly prefer the underlying asset to be more risky.

2 The Model

A seller/issuer (she) has a project or asset that generates a random return with outcomes in the interval $X = [0, \overline{x}] \subseteq \mathbb{R}_+$. The project's return is governed by the distribution $H : X \to [0, 1]$. We assume that the random return has a finite expectation. The seller has no cash, and needs to raise funds of $c \in (0, 1)$ in order to finance the project.

Risk Preferences and Budgets of Investors There is a unit mass of potential, risk-averse investors/buyers/agents (he). Each one of them is described by a limited budget that is normalized here to 1, and by a preference relation that can be represented by a dual utility function (see Yaari [1987]): Let $\mathcal{V}$ be the set of random variables with outcomes in the interval X, defined on a given non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cumulative distribution function of a random variable $v \geq 0$ is denoted by H_v. Let $g : [0,1] \to [0,1]$ be increasing with $g(0) = 0$ and $g(1) = 1$. The functional defined by

$$\mathcal{U}_g(v) = 0 + \int_0^{\overline{x}} g(1 - H_v(s))ds$$

for each $v \in \mathcal{V}$ is called Yaari's *dual utility* with distortion function g. Utility here is specified in monetary units, and $\mathcal{U}_g(v)$ is the certainty equivalent of lottery v.

Risk aversion in the standard sense of aversion to mean preserving spreads (second-order stochastic dominance) is here equivalent to the convexity of the distortion function g. Note that integration by parts yields[11]

$$\mathcal{U}_g(v) = 0 + \int_0^{\overline{x}} g(1 - H_v(s))ds = \int_0^{\overline{x}} g'(1 - H_v(s))\, s\, dH_v(s)\,.$$

In other words, dual utility modifies the standard expectation operator $\mathbb{E}_H[v] = \int_0^{\overline{x}} s\, dH_v(s)$ by adding a set of weights that depend on the cumulative probability of an outcome. Each outcome s is weighted by the weight $g'(1 - H_v(s))$ that is decreasing if g is convex, i.e., if the agent is risk-averse. Thus, standard risk aversion is created here by having higher weights on less favorable outcomes - this is also the main appeal from the perspective of risk measures. The ubiquitous channel that creates risk aversion - decreasing marginal utility of money - is absent and the marginal utility of money is constant. Risk neutrality corresponds to the distortion g being the identity function, in which case we obviously have

$$\mathcal{U}_g(v) = \int_0^{\overline{x}} s\, dH_v(s) = \mathbb{E}[v].$$

Private Information We assume that the risk preference of each investor is their private information, and described by their type θ which determines the distortion function g_θ. We assume that there are two types of investors: l types (*conservative* investors) with low risk-tolerance and h types (*aggressive* investors) with high risk-tolerance $\theta \in \Theta = \{l, h\}$. Each type occurs with probability $f_l, f_h > 0$, respectively, such that $f_h + f_l = 1$.[12]

[11] The integral here is in the Lebesgue-Stieltjes sense.

[12] The two type model is sufficient to derive the basic structure of an asset tranched into senior debt, junior debt, and equity. With more risk types, these basic securities will be subdivided further into

By the representation of dual utility, the function g_θ is increasing and satisfies $g_\theta(0) = 0$ and $g_\theta(1) = 1$ for each $\theta \in \Theta$. Additionally, we assume that $g_\theta(p)$ is convex, that corresponds to assuming that all investors are risk averse (i.e. averse to mean-preserving spreads). We further assume that the investors' risk attitudes are ordered: g_l is a convex transformation of g_h, meaning that conservative type l investors are more risk-averse than the aggressive type h investors. It is noteworthy that this also implies here that $g_l(p) \leq g_h(p)$ for all $p \in [0,1]$, with strict inequality holding for some p.

We first assume, for simplicity, that the issuer is risk neutral. We will discuss the extension to a risk-averse issuer in Section 6. Finally, to rule out trivial cases, we assume that it is technically feasible to raise the necessary funds c from investors. Concretely, we assume that the conservative investors - who require a higher premium in order to purchase risk - value the project higher than its cost:

$$\int_X g_l(1 - H(x))dx > c.$$

As both types of investors are risk-averse, the above condition also implies that the project's expected return is at least c.

Before concluding this section, we note that having the same budget for both types is for illustrative convenience only. Our results generalize in a straightforward manner to the case where budgets are heterogeneous and are the agents' private information (see Section 7 for details).

2.1 Risk Measures and Dual Risk-Preferences

A prominent example of the class of risk preferences we consider correspond to coherent distortion risk measures or spectral risk measures.[13] Given a function $\hat{g} : [0,1] \to [0,1]$, a distortion risk measure assigned to a random variable v is defined via a Choquet integral as (see, e.g. Ruschendorf [2013]):

$$\int_{-\infty}^{0} 1 - \hat{g}(1 - F_{-v}(s))ds - \int_{0}^{\infty} \hat{g}(1 - F_{-v}(s))ds$$

where $F_{-v}(s) = \mathbb{P}[-v \leq s]$. Simple algebra shows that such a distortion risk measure equals the minus of the Yaari dual utility with $g(x) = 1 - \hat{g}(1 - x)$. It is well known that a distortion risk measure is *coherent* if and only if $\hat{g}$ is concave, i.e. g is convex

tranches, one for each type. Since adding more investor types provides limited additional economic insight, we assume two types to simplify the exposition and focus on the economically novel aspects of the model.

[13]In the financial literature on risk measures dual utility is sometimes called *monetary utility*.

((see Arztner et al. [1999]).[14] Special cases of distortion risk measures are:

Value at Risk Value at risk is a commonly used risk-measure and defined as

$$VaR(v) = \sup\{s \colon \mathbb{P}[v > s] \geq 1 - \alpha\}.$$

It represents the threshold s where the likelihood of achieving a return greater than s is $1 - \alpha$. This risk measure corresponds to the distortion

$$g^{\alpha}_{VaR}(p) = \begin{cases} 1 & \text{if } p \geq 1 - \alpha \\ 0 & \text{else} \end{cases}$$

where $\alpha \in [0, 1]$. Value at risk is not coherent, and hence its use is not anymore recommended by the Basel framework.

Expected Shortfall The main alternative is *expected shortfall* that constitutes a coherent risk measure. It refines the value at risk by considering the expected return conditional on it lying below the level that is exceeded with probability $1 - \alpha$

$$ES(v) = \mathbb{E}[v | v \leq VaR(v)]\,.$$

The expected shortfall corresponds to the distortion

$$g^{\alpha}_{ES}(p) = \frac{1}{\alpha} \max\{p - (1 - \alpha), 0\}\,.$$

where $\alpha \in [0, 1]$.

Exponential Distortion Risk Measure Another very popular ordered class of coherent risk measures corresponds to the family of *exponential distortions*:

$$g^{\alpha}(p) = \frac{e^{-\alpha(1-p)} - e^{-\alpha}}{1 - e^{-\alpha}},$$

where $\alpha \in [0, 1]$.

Remark: When limits on value at risk, expected shortfall, or another risk measure are imposed on a financial decision maker, these enter her decision problem via a constraint. This arises, for example, if the investor is risk neutral but the set of feasible investments is constrained by a lower bound on their expected shortfall, e.g., due to regulation such as the one following Basel III and IV. Practically, this implies (via a

[14] A risk measure is coherent if it is monotonic, positively homogeneous and subadditive.

Lagrange multiplier approach) the maximization of a weighted sum of the expected value and the expected shortfall, e.g. $(1-\lambda)\mathbb{E}[v] + \lambda ES(v)$. For example in the case of expected shortfall, the corresponding distortion function is then given by

$$g_\lambda^\alpha(p) = (1-\lambda)p + \frac{\lambda}{\alpha}\max\{p-(1-\alpha),0\}\,.$$

Loss Averse Preference Finally, a simple and well-known example of an ordered family of distortions appearing in behavioral economics are the *loss averse* preferences with linear local utility, studied by Kőszegi and Rabin [2006] and by Masatlioglu and Raymond [2016]. These correspond to the distortion function

$$g^k(p) = kp^2 + (1-k)p\,.$$

Here $k \in [0,1]$ captures the degree of risk-aversion: $k=0$ yields risk neutrality, while $k=1$ yields the highest risk aversion in this class.

3 Mechanisms: Menus of Securities

We restrict attention to deterministic direct mechanisms. Note that this is **not** without loss of generality: random mechanisms can do here better. While adding even more risk in order to extract funds from risk-averse agents seems rather counterintuitive, randomization can sometimes help with their screening.[15] Nevertheless, stochastic mechanisms are rarely, if ever, used in practice - it is very hard to credibly commit to the announced randomizations - and we abstract from them here.

Since we have a continuum of agents, we look at mechanisms $(R_\theta, t_\theta)_{\theta\in\Theta}$ consisting of a menu of *asset-backed securities* R_l, R_h and their prices t_l, t_h. For each θ, $R_\theta(x)$ is the payoff of the security R_θ if the underlying assets return equals x and $t_\theta \geq 0$ is the *price* of this security.

We restrict attention to monotonic contracts having non-negative returns. That is, for any θ and x, $R_\theta(x) \geq 0$ and $R_\theta(x)$ is non-decreasing in x. We also assume that, for each θ, the payoff of the security $R_\theta(x)$ is absolutely continuous in the project's return, and we denote by $R'_\theta(x)$ its generalized derivative. In addition, the promised return of all offered asset-backed securities cannot exceed the value of the underlying asset. That is, for any $x \in X$, the following feasibility constraint must hold:

$$\sum_{\theta\in\{l,h\}} f_\theta\, R_\theta(x) \leq x.$$

[15] For an example in the context of optimal insurance see the online Appendix A of Gershkov et al. [2022]).

It directly follows that $R_\theta(0) = 0$ for all θ.

Fix any mechanism $(R_\theta, t_\theta)_{\theta\in\Theta}$. An agent with type θ who reports to be of type θ' obtains dual utility

$$U(\theta, \theta') = -t(\theta') + \int_X R'_{\theta'}(x) g_\theta(1 - H(x))dx.$$

With a slight abuse of notation, we let $U(\theta) = U(\theta, \theta)$ denote a type-θ agent's utility when he and all other agents report truthfully.

Remark: Payments made to investors in the context of asset-based securities are exclusively sourced from the asset's returns. Alternatively, one can extend the model by allowing the issuer to add the capital collected from investors in order to increase some payments. This modifies the feasibility constraint to:

$$\sum_{\theta\in\{l,h\}} f_\theta\, R_\theta(x) \le x + \sum_\theta f_\theta t_\theta - c.$$

However, as demonstrated at the end of the Appendix, it is never strictly optimal for the issuer to adopt this re-investment approach. It is thus without loss of generality to focus on asset-based securities.

3.1 Implementable Mechanisms

As the conservative l type of investor assigns a strictly higher value to the project than its cost, the issuer can always offer a single security that will be bought by all types, and that has a price greater than c. Doing so yields a strictly positive profit, so it cannot be optimal not to finance the project. This yields that, in any optimal menu, it holds that the sum of the security prices exceeds the financing cost c

$$f_h t_h + f_l t_l \ge c. \tag{FC}$$

Therefore, we only consider below mechanisms in which the project is successfully financed, and thus (FC) needs to hold. For a type θ agent not to deviate and claim to be of type θ', it needs to hold that

$$-t_\theta + \int_X R'_\theta(x) g_\theta(1 - H(x))dx \ge -t_{\theta'} + \int_X R'_{\theta'}(x) g_\theta(1 - H(x))dx\,.$$

This is the same as having the difference in the security equivalents of the assets exceed the difference in their prices:

$$\int_X \left[R'_\theta(x) - R'_{\theta'}(x)\right] g_\theta(1 - H(x))dx \geq t_\theta - t_{\theta'} \tag{IC-θ}$$

Similarly, in order to make a type θ agent purchase the security offered to him instead of pursuing an outside option (e.g., acquiring a risk-free government bond with a fixed interest rate) that is normalized here to yield zero utility, it must be the case that

$$\int_X R'_\theta(x) g_\theta(1 - H(x))dx \geq t_\theta \tag{IR-θ}$$

The feasibility constraint requires that the promised payments from the asset cannot exceed the value of the asset: for each $x \in X$

$$f_l\, R_l(x) + f_h\, R_h(x) \leq x$$

that is equivalent to $R_\theta(0) = 0$ for $\theta \in \Theta$ and

$$\int_0^x [f_l R'_l(z) + f_h R'_h(z)]dz \leq x \tag{Feasibility}$$

for all $x > 0$.

In addition, recall that we require

$$R'_\theta(x) \geq 0 \tag{M}$$

for $\theta \in \Theta$ and for all x, and that each type θ has a limited budget of 1:

$$t_\theta \leq 1. \tag{BC}$$

4 Extreme Securities

For the derivation of the optimal menu of securities, we first describe a fundamental result about the best and worst securities for any risk-averse investor who selects one among the set of doubly monotonic securities that satisfy an iso-cost condition for the issuer. The argument has been mainly developed in the insurance literature, and we transfer it here to the security design setting.

We say that a security R is *double monotonic* if, in addition to R being monotonic in the asset's return, the part of the asset left with seller $x - R(x)$ is also monotonic in x. Let $\mathcal{R}_\gamma$ denote the set of all feasible, doubly monotonic securities which lead to the same expected payment γ

$$\mathcal{R}_\gamma = \{R \colon \mathbb{E}[R(x)] = \gamma\}.$$

Let x^d be a solution to $\mathbb{E}[\min\{x, x^d\}] = \gamma$ and note that

$$R^\gamma_{debt}(x) = \min\{x, x^d\}$$

represents a *debt* contract in $\mathcal{R}_\gamma$. Similarly, let x^o be a solution to $\mathbb{E}[(x - x^o)_+] = \gamma$ and note that

$$R^\gamma_{equity}(x) = (x - x^o)_+$$

is an *equity* contract included in $\mathcal{R}_\gamma$.

Theorem 1 *Consider any security $R \in \mathcal{R}_\gamma$. Then it holds that*

$$R^\gamma_{equity}(x) \preceq R(x) \preceq R^\gamma_{debt}(x)$$

where $\preceq$ denotes second-order stochastic dominance.

In other words, a debt security (an equity) is the least (most) variable security among the securities with a given cost, and therefore the best (worst) choice for **any** risk-averse investor. The right side is well-known, and only requires the monotonicity of the relevant securities (see, for example, Van Heerwaarden et al. [1989]).[16] The left side is proved in Gershkov et al. [2022] and requires double monotonicity.[17]

5 Optimal Security Design

The profit of the issuer, when financing the project via the mechanism $(R_\theta(x), t_\theta)_{\theta\in\Theta}$, is given by:

$$\mathbb{E}_H[x] - \int_X [f_h R'_h(x) + f_l R'_l(x)]\,(1 - H(x))dx + [f_h t_h + f_l t_l - c]\,.$$

Our first Lemma demonstrates that the issuer never wants to raise more than c: raising funds that are not strictly needed is too costly in terms of the foregone returns that could be obtained by retaining a higher portion of the underlying asset. This

[16]It generalizes famous results by Arrow [1963] and by Borch[1960] who showed that deductibles lead to the lowest variance among all contracts with the same cost.

[17]Should we only impose monotonicity, then the "live-or-die" contract, as investigated by Innes [1990], emerges as the least preferred asset by any risk-averse agent in such context. This contract, given by $R(x) = x\mathbf{1}_{x\geq k}$, which essentially transfers the asset's ownership if its return surpasses a threshold k, does not satisfy double monotonicity, and is thus different from an equity.

happens because all investors demand here a risk premium in order to acquire risk, while the seller is risk-neutral.

Lemma 1 *Fix any optimal menu* $(R_\theta, t_\theta)_{\theta\in\Theta}$. *It must hold that*

$$f_h t_h + f_l t_l = c. \tag{FC'}$$

By the above Lemma, and since the asset's expected return $\mathbb{E}_H[x]$ is fixed, the seller's objective function reduces to:

$$\min_{(R_\theta(x),t_\theta)_{\theta\in\Theta}} \left\{ \int_X \left[f_h R_h'(x) + f_l R_l'(x)\right](1 - H(x))dx \right\}$$

subject to constraints (IR), (IC), (Feasibility), (M), (BC) and (FC').

In words, the issuer wants to minimize the loss of potential cash-flow from the asset caused by the sale of securities to investors, subject to the constraints that she needs to raise a sum c from them, and that the mechanism is implementable.

We next derive the optimal security design, distinguishing between two cases. In the first case, the project can be entirely financed by selling securities solely to aggressive investors. The solution to the first case offers a building block for the second, more complex case, where it is necessary to sell securities to both types of investors.

5.1 Aggressive Investors are Sufficient to Finance the Project

Suppose first that the project can be financed by raising money only from the high types, i.e. $c \leq f_h \leq 1$. In this case, the most efficient way of raising the necessary funds is to offer securities only to the aggressive investors, so that it is optimal to set $t_h = \frac{c}{f_h}$ and $t_l = 0$. An alternative interpretation of this basic problem is one where a single investor has a high enough budget to finance the entire project, as in a specially tailored, single-tranche collateralized debt obligation (CDO). The resulting maximization problem for the issuer of the security becomes:

$$\begin{aligned}
\text{(Problem P)} \quad & \min_{R_h} \left\{ f_h \int_X R_h'(x)(1 - H(x))dx \right\} \\
\text{s.t.} \; & \int_X R_h'(x) g_h (1 - H(x))dx = \frac{c}{f_h} \\
& R_h'(x) \geq 0, && \forall x \in X \\
& f_h \int_0^x R_h'(z)dz \leq x, && \forall x \in X \\
& R_h(0) = 0
\end{aligned}$$

The first equality in the above set of constraints reflects the binding participation constraint for the aggressive types who exhaust their budget. This is an *isoperimetric* constraint. The second inequality is the *monotonicity* constraint. The last two constraints ensure that the payouts to the agents do not exceed the return of the security (i.e. the feasibility constraint) where the second last constraint is a *majorization* constraint, and the last constraint is a *boundary condition*.

The convexity of the function g_h is equivalent here to assuming that the function $z \mapsto \frac{z}{g_h(z)}$ is decreasing, or that the function $x \mapsto \frac{1-H(x)}{g_h(1-H(x))}$ is increasing. Intuitively, when this last condition holds, it is beneficial for the seller to make $R_h'(x)$, the security's slope, as large as the majorization constraint allows for small realizations of the asset's return x, and as small as possible for larger realizations obtained as soon as the isoperimetric constraint is satisfied. Instead of using this intuition for a direct proof, we use below the right-hand side of Theorem 1 (recall that this part does not use double-monotonicity, and therefore this assumption is not invoked here).

Proposition 1 *Let x^* denote the solution to*

$$\int_0^{x^*} g_h(1 - H(z))dz = c$$

and note that $c < x^* < \bar{x}$.[18] *The optimal security is given by*

$$R_h^*(x) = \begin{cases} \frac{x}{f_h} & \textit{for } x \leq x^* \\ \frac{x^*}{f_h} & \textit{otherwise} \end{cases}$$

and by $t_h^* = \frac{c}{f_h}$. That is, a debt contract with interest rate $\frac{x^*}{c} - 1 > 0$ is optimal.

Theorem 1 implies that, among securities with a given cost, a debt contract yields the highest utility to the risk-averse agent. Equivalently, to accord a fixed utility to such an agent — as necessitated by (IR-l) and the financing need — the most cost-effective strategy for the issuer is to proffer a debt contract. It is worth noting that this conclusion applies to any risk-averse agents, regardless of whether their preference aligns with EU or non-EU models.

The above proposition also implies that the optimal security's interest rate depends on the risk-aversion of the aggressive investors, on the project's return profile, and on the financing cost c.

Observation: The above result includes several very intuitive predictions that can be tested empirically: ceteris paribus, a better distribution of returns H (in the sense

[18]This exists because, by assumption $\int_X g_h(1 - H(x))dx \geq c$ so that the project can be financed at all.

of first order stochastic dominance) yields a lower interest rate; a higher degree of risk-aversion yields a higher interest rate; a higher financing need c also yields a higher interest rate.[19]

5.2 Both Types of Investor are Needed to Finance the Project

Consider now the case where both types of investor are needed to finance the project, i.e. $c > f_h > 0$. For this setting, the participation constraint of a low type, (IR-l), must be binding. If not, then the seller can increase her profit by extracting higher payments from both types. Analogously, the incentive constraint of a high type, (IC-h), must also bind.

We first derive the optimal mechanism for the relaxed problem where we do **not** impose the incentive constraint for a low type (IC-l), nor the participation constraint for a high type, (IR-h). We later check that the obtained solution to the relaxed problem indeed satisfies these omitted constraints. Formally, the relaxed problem is:

$$\text{(Problem Q)} \quad \min_{(R_\theta, t_\theta)_{\theta\in\Theta}} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)](1 - H(x))dx \right\}$$
$$\text{s.t.} \int_X R_l'(x) g_l (1 - H(x))dx = t_l$$
$$\int_X \left[R_h'(x) - R_l'(x)\right] g_h (1 - H(x))dx = t_h - t_l$$
$$f_l t_l + f_h t_h = c$$
$$R_h'(x), R_l'(x) \geq 0, \ \forall x \in X$$
$$f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x, \ \forall x \in X$$
$$R_h(0) = R_l(0) = 0$$

The first equality is the (IR-l) constraint, the second one is (IC-h), the third line is (FC'), the last two lines ensure feasibility. The following Lemma demonstrates that, in any solution to the relaxed problem, all aggressive, high types invest their whole budget, i.e. $t_h = 1$.

Lemma 2 *Suppose that $(R_\theta, t_\theta)_{\theta\in\Theta}$ is a solution to (Problem Q). If $t_l > 0$, then $t_h = 1$.*

It follows from the above lemma that, in the optimal mechanism, the seller raises f_h (in total) from the aggressive investors and still needs to raise $f_l t_l = c - f_h$ from

[19] To see the last point, vary c and let $x(c)$ be a solution to $\int_0^{x(c)} g_h(1 - H(z))dz = c$. Taking the derivative with respect to c twice in the above expression yields that: x is convex and hence the corresponding interest rate, $\frac{x(c)}{c} - 1$, is increasing in the financing need c.

the conservative investors. That is, $t_l = \frac{c-f_h}{f_l}$. Given this insight, the relaxed problem Q can be further simplified into:

$$
\begin{aligned}
\text{(Problem } Q'\text{)} \quad & \min_{R_l, R_h} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)](1-H(x))dx \right\} \\
& \text{s.t.} \int_X R_l'(x) g_l (1-H(x))dx = \frac{c-f_h}{f_l} \\
& \int_X [R_h'(x) - R_l'(x)] g_h (1-H(x))dx = t_h - t_l = \frac{1-c}{f_l} \\
& R_h'(x), R_l'(x) \geq 0, \ \forall x \in X \\
& f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x, \ \forall x \in X \\
& R_h(0), R_l(0) = 0,
\end{aligned}
$$

Below, we outline the solution to (Problem Q'), which also serves as a solution to the original problem.

Theorem 2 *Suppose that $c > f_h$ and let x_l^*, x_h^* denote the solutions to:*

$$
\int_0^{x_l^*} g_l (1-H(x))dx = c - f_h
$$

and

$$
\frac{1}{f_h} \int_{x_l^*}^{x_h^*} g_h (1-H(x))dx = \frac{1}{f_l} \int_0^{x_l^*} g_h (1-H(x))dx + \frac{1-c}{f_l}
$$

respectively. The optimal menu of securities is given by $t_l^* = \frac{c-f_h}{f_l}$, $t_h^* = 1$, and by

$$
R_l^*(x) = \begin{cases} \frac{x}{f_l} \textit{ for } x \leq x_l^* \\ \frac{x_l^*}{f_l}, \textit{ otherwise} \end{cases}
$$

$$
R_h^*(x) = \begin{cases} 0, \textit{ for } x \leq x_l^* \\ \frac{x - x_l^*}{f_h}, \textit{ for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^* - x_l^*}{f_h}, \textit{ otherwise} \end{cases}
$$

The proof of the Theorem can be found in the Appendix. It establishes that the optimal mechanism is a menu of two contracts:

1. *Senior debt* with interest rate

$$
\frac{\frac{x_l^*}{f_l}}{t_l^*} - 1 = \frac{\frac{x_l^*}{f_l}}{\frac{c-f_h}{f_l}} - 1 = \frac{x_l^*}{c - f_h} - 1
$$

   that is determined by the participation constraint of the conservative investors (i.e. (IR-l)).

2. *Junior debt* with interest rate

$$\frac{\frac{x_h^*-x_l^*}{f_h}}{t_h^*}-1=\frac{x_h^*-x_l^*}{f_h}-1,$$

that is determined by the incentive compatibility constraint of the aggressive investors (i.e. (IC-h)).

Furthermore, the amount of money that any agent can invest in senior debt is limited to $\frac{c-f_h}{f_l}$.

To gain some intuition on why this "waterfall" structure is optimal, note first that a similar structure of securities (i.e., senior/ junior debt and equity) is also optimal under complete information if investors have dual preferences. With complete information, the designer only needs to induce an efficient risk sharing among agents (subject to participation constraints). As the investors' risk preferences are ordered, the efficient way to share risk is to allocate the safest possible asset (i.e., senior debt) to the most risk-averse investor type, to allocate the second safest possible asset (i.e., junior debt) to the more risk tolerant investors, and to keep the remaining, most risky asset (i.e., equity). In this case, the interest rate for the senior debt is still determined by the (IR-l) constraint, and it is equal to that in the incomplete information setting. But, under complete information, the interest rate for junior debt is determined by the (IR-h) constraint, and it is lower than that under incomplete information where it is determined by the (IC-h) constraint.

The optimal allocation under incomplete information presents an atypical structure compared to the general class of screening models. Recall that in the standard screening problem (see, for example, Mussa and Rosen [1978] and Maskin and Riley [1984]), the allocation designed for the low type is often distorted downwards from its efficient level in order to reduce the information rent extracted by the high type. This is **not** the case here! As illustrated above, under incomplete information, conservative investors (low types) obtain exactly the same product and pay the same price as in the complete information benchmark. By contrast, the interest rate offered to the aggressive investors (high types) is distorted upwards to meet their incentive compatibility constraint. This happens here because the efficient risk allocation to the low types - senior debt with the minimal interest rate to guarantee their participation - happens to also minimize the high type's information rent. Since the high type earns such a rent because he values the risky asset more than the low type, the information rent is minimized when the low type receives the safest asset, i.e., in an allocation that coincides with the solution to the complete information benchmark. In other words, the most cost-effective way to provide the high type the required information rent is to maintain the waterfall structure that is optimal under complete information while increasing the interest rate for junior debt. We will further illustrate this point in Section 6.

Remark (More Investor Types): The construction outlined above can easily be generalized to accommodate a setting with more than two types of risk-averse investors. As the required capital c increases, the involvement of additional, increasingly conservative investors becomes necessary for financing. Consequently, the issuer releases more tranches, corresponding to each investor type. Notably, our model clearly predicts that larger issues consist of more tranches, a feature often observed in reality.

Remark (Pooling Assets is Optimal): Suppose that originally, there were two separate security issues backed by assets x and y with costs c_x and c_y, respectively. Then, pooling the two underlying assets and providing a security backed by the pooled assets is always beneficial for the issuer. Indeed, the least costly way to finance the project $x+y$ with total cost c_x+c_y is to issue tranched, optimal debt contracts backed by it (as discussed above). The original securities, independently backed by x and y and potentially optimal in their own right, do not generally add up to such contracts unless x and y are comonotonic random variables. It then directly follows that, from the issuer's perspective, the foregone return induced by the separate security issues must be higher than that induced by the combined, optimally-structured issue.

Remark (Expected Utility): Even under complete information, a "waterfall" structure does not emerge as optimal when investors have expected utility preferences. This happens because, in such cases, the marginal utility of money is not constant, leading to a much more complex solution to the induced risk-sharing problem (see, for example, Allen and Gale [1989]).

6 Comparative Statics

In this section we present several comparative statics results. These results are markedly different from those obtained in the same model under complete information.

We first investigate the effects of having a better/safer asset in the sense of a first order stochastic dominance(FOSD) or second order stochastic dominance(SOSD) shift, respectively. We then discuss the effect of a decrease in the financing cost. We focus here on the more interesting case where both types of investor are needed for financing the project, e.g. assume $f_h < c$.

To simplify notation let

$$i_l^x = \frac{x_l^*}{c - f_h} - 1$$
$$i_h^x = \frac{x_h^* - x_l^*}{f_h} - 1$$

denote the interest rates offered by the issuer to the conservative and aggressive investors, respectively, in the optimal menu of asset backed securities.

6.1 Who Prefers a Better or a Safer Asset?

Our main finding here is that the issuer always prefers a stochastically better/safer asset, while the aggressive investors always prefer a worse/riskier asset. Recall that H denotes the distribution of returns of the asset.

Proposition 2 *Either a SOSD or FOSD shift of H results in:*

1. *A decrease in the interest rate i_l for conservative investors.*
2. *A decrease in the expected cost of financing $\int_0^{x_h^*}(1 - H(z))dz$.*
3. *A decrease of the surplus $\int_{x_l^*}^{x_h^*} g_h(1 - H(z))dz - 1$ obtained by the aggressive investors.*

Furthermore, an FOSD shift of H leads to a decrease in the interest i_h obtained by aggressive investors.

The above proposition shows that, with a stochastically better/safer asset, the issuer offers a lower interest rate to conservative investors, gives up a smaller share of the asset, and leaves less information rent to the aggressive investors. With a stochastically better asset, the interest rate offered to aggressive investors will also decrease. Intuitively, the comparative advantage of aggressive investors lies in their higher tolerance for risk. If the asset becomes less risky this advantage becomes less relevant and the rents of aggressive investors decrease as they can be more easily substituted by conservative investors.

6.2 The Project's Cost

We find that, as the project becomes less costly to implement, both the offered interest rates and the financing cost decrease. Conservative investors are indifferent, but the aggressive investors are worse-off.

Proposition 3 *In the optimal contract that finances asset x and raises c, the following hold:*

1. *Both interest rates, i_l and i_h, increase in c;*
2. *Aggressive investors prefer to finance costlier projects since their utility, given by*

$$\int_{x_l^*}^{x_h^*} g_h(1 - H(z))dz - 1$$

increases in c.

As the cost of the project increases, the issuer needs to forgo a larger share of the security to raise sufficient funds. Consequently, the riskiness of both senior and junior debt rises, necessitating higher interest rates. Moreover, as the asset sold to the conservative investors becomes riskier, the aggressive investors can capture a higher information rent.

7 Extensions

In this section we offer several extensions to our basic model: We first analyze a model where the investors' budgets are heterogeneous and private information (thus types are here two-dimensional). We next allow for a risk-averse issuer. Lastly, we consider an extension where the designer is not allowed to impose purchasing limits on buyers. In all extensions, we focus on the more interesting case where both risk types are needed to finance the project.

7.1 Private Budgets

We consider here agents who have two possible types of budgets: $b = \beta < 1$ with probability p and $b = 1$ with probability $1 - p$. The individual budgets are privately known, and, for each agent, the perceived distribution of his budget is independent of the distribution of the agent's risk type. Under this assumption, the analysis for more budget types remains essentially the same. For later use, we also define the average budget as $\bar{\beta} = p\beta + 1 - p$.

Let $x_{l,1}^*$ denote the solution to

$$\int_0^{x_{l,1}^*} g_l(1 - H(x))dx = c - f_h\bar{\beta}$$

and let $x^*_{h,1}$ denote the solution to

$$\frac{1}{f_h\bar{\beta}}\int_{x^*_{l,1}}^{x^*_{h,1}} g_h(1-H(x))dx = \frac{1}{f_l\bar{\beta}}\int_0^{x^*_{l,1}} g_h(1-H(x))dx + \frac{\bar{\beta}-c}{f_l\bar{\beta}}$$

Theorem 3 *If $c > f_h\bar{\beta}$, the menu of securities described below is optimal:*

$$R^*_{l,1}(x) = \begin{cases} \frac{x}{f_l\bar{\beta}} \text{ for } x \le x^*_{l,1} \\ \frac{x^*_{l,1}}{f_l\bar{\beta}}, \text{ otherwise} \end{cases}$$

$$R^*_{h,1}(x) = \begin{cases} 0, \text{ for } x \le x^*_{h,1} \\ \frac{x-x^*_{l,1}}{\bar{\beta}f_h}, \text{ for } x^*_{l,1} \le x \le x^*_{h,1} \\ \frac{x^*_{h,1}-x^*_{l,1}}{\bar{\beta}f_h}, \text{ otherwise} \end{cases}$$

$$R^*_{l,\beta}(x) = \beta R^*_{l,1}(x) \text{ and } R^*_{h,\beta}(x) = \beta R^*_{h,1}(x) \text{ for all } x$$

Moreover, the price of securities are given by $t_{l,1} = \frac{c-f_h\bar{\beta}}{f_l\bar{\beta}}$, $t_{h,1} = 1$, $t_{l,\beta} = \beta t_{l,1}$, $t_{h,\beta} = \beta t_{h,1}$.

It is still optimal for the issuer to offer senior debt to the conservative investor with a high budget of 1, and junior debt to the aggressive type with a high budget 1. However, the issuer now introduces an additional option for the more budget-constraint investors: these individuals can purchase a share $\beta < 1$ of the corresponding debt at a share β of the price charged to high-budget investors.

The details of the proof can be found in the appendix, but we provide an outline below: We first derive the optimal mechanisms for the scenario in which agents have heterogeneous yet publicly known budgets (while risk preferences remain private to the agents). Next, we demonstrate that this mechanism remains implementable even when the budget also constitutes private information of the agents. Thus, it must be optimal in this context as well. This discovery underscores the notion that, in a large market, small investors do not derive any informational advantage or rent by keeping their budget information private.

7.2 Security Design by a Risk-Averse Issuer

We now consider the extension in which the issuer is also dual risk-averse, and her dual utility is represented by a convex distortion function g. We assume here that the issuer is less risk-averse than the conservative investors, i.e., g is less convex than g_l.[20]

[20] If the issuer is more risk-averse than the conservative investors, then the maximization problem is slightly different because the seller will want to extract all the cash from investors. However, this new problem can be solved in an analogous manner.

We say that a menu of securities, $(R_\theta)_{\theta=l,h}$ is *double monotonic* if, in addition to R_θ being monotonic for all θ, the function

$$R(x) = x - \sum_{\theta=l,h} f_\theta R_\theta(x)$$

is also monotonic, i.e., the issuer's own tranche is also monotonic in the asset's return. This assumption, commonly adopted in the finance literature, serves as a safeguard against issuers' potential manipulation of asset cash flows — either through external lending or inter-project transfers — with the intent of diminishing payouts to investors. It directly follows from double monotonicity that $f_l R_l'(x) + f_h R_h'(x) \leq 1$ almost everywhere on X.

Restricting attention to doubly monotonic contracts, and following essentially the same steps as above, the issuer's problem becomes

$$\begin{aligned}
&\min_{R_l, R_h} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)] g(1 - H(x)) dx \right\} \\
&\text{s.t.} \int_X R_l'(x) g_l(1 - H(x)) dx = \frac{c - f_h}{f_l} \\
&\qquad \int_X [R_h'(x) - R_l'(x)] g_h(1 - H(x)) dx = t_h - t_l = \frac{1 - c}{f_l} \\
&\qquad R_h'(x), R_l'(x) \geq 0 \ \forall x \in X \\
&\qquad f_l R_l'(x) + f_h R_h'(x) \leq 1 \ \forall x \in X \\
&\qquad R_h(0), R_l(0) = 0
\end{aligned}$$

The third and fourth constraints represent the double monotonicity conditions. Together, the two conditions imply that the contract is feasible, and thus the feasibility constraint $f_l R_l(x) + f_h R_h(x) \leq x$ for all x is no longer needed.

If the issuer is also less risk-averse than the aggressive investors, i.e., if g is less convex than g_h, then our previous analysis for a risk-neutral issuer applies. Novel findings arise for the case where the issuer is more risk-averse than the aggressive investors, i.e., if g is more convex than g_h. Thus, we focus below on the case where the seller's risk aversion is intermediate between those of the two types of investors.

Proposition 4 *Suppose that $c > f_h$ and that g is more convex than g_h. Let $\tilde{x}_l$, $\tilde{x}_h$ denote the solutions to*

$$\int_0^{\tilde{x}_l} g_l(1 - H(x)) dx = c - f_h$$

and

$$\frac{1}{f_h} \int_{\tilde{x}_h}^{\bar{x}} g_h(1 - H(x)) dx = \frac{1}{f_l} \int_0^{\tilde{x}_l} g_h(1 - H(x)) dx + \frac{1 - c}{f_l}$$

respectively. The optimal menu is given by $\tilde{t}_l = \frac{c-f_h}{f_l}$, $\tilde{t}_h = 1$, *and*

$$\tilde{R}_l(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \leq \tilde{x}_l \\ \frac{\tilde{x}_l}{f_l} \text{ otherwise} \end{cases}$$

$$\tilde{R}_h(x) = \begin{cases} 0 \text{ for } x \leq \tilde{x}_h \\ \frac{x-\tilde{x}_h}{f_h} \text{ otherwise} \end{cases}$$

A conservative investor still receives a senior debt contract with interest rate $\frac{\tilde{x}_l}{c-f_h}$, but an aggressive investor now receives the remaining equity after all debt-holders have been paid. It directly follows that the part of the asset that is kept by the designer, $x - f_l R_l(x) + f_h R_h(x)$, now takes the form of junior debt, and is given by:

$$R^*(x) = \begin{cases} 0 \text{ for } x \leq \tilde{x}_l \\ x - \tilde{x}_l \text{ for } \tilde{x}_l < x \leq \tilde{x}_h \\ \tilde{x}_h - \tilde{x}_l, \text{ otherwise} \end{cases}$$

7.3 The Optimal Mechanism without Purchasing Limits

In this section, we characterize the optimal mechanism for the case where both types are needed to finance the project, and where purchasing limits, as featured in the benchmark model, cannot be imposed. While in the main part of the paper we explored the optimal mechanism design problem where the issuer fully controls both the allocation and monetary transfer of every type, here we consider a simpler and more prevalent selling procedure where every investor decides how many units of each security to acquire. The main change lies in the incentive constraint of the aggressive investor type who must now be deterred from buying multiple units of the security intended for the conservative types. This change will influence the offered interest rates, but it will not alter the foundational waterfall structure of the optimal menu of contracts.

For an alternative motivation, note that if agents can trade among themselves after the completion of the initial issue, aggressive, less risk-averse investors may be able to purchase securities from the conservative, more risk-averse investors at a mutually beneficial price. Foreseeing this possibility, the aggressive investors will refrain from buying at the initial issue, causing a loss of revenue to the issuer. The incentive constraint imposed in this section is precisely constructed in order to avoid this possibility - thus, there will be no incentives for trading among the various investors after the initial issue.

As above, we first consider the relaxed problem where we only impose the (IR-l) and (IC-h) constraints together with the feasibility constraints, and then verify the solution also satisfies the omitted constraints.

In any implementable mechanism where both types are needed to finance the project, an aggressive investor derives strictly positive utility from pretending to be a conservative type. As Yaari's dual utility is homogeneous, the aggressive type's best deviation payoff is to purchase $\frac{1}{t_l}$ units of R_l. (IR-l) is then still given by

$$\int_X R_l'(x) g_l(1-H(x))dx = \frac{c-f_h}{f_l}$$

but (IC-h) changes to

$$\int_X \left[R_h'(x) - \frac{f_l}{c-f_h} R_l'(x)\right] g_h(1-H(x))dx = t_h - t_l \times \frac{1}{t_l} = 0.$$

Theorem 4 *Suppose that $c > f_h$ and let $\hat{x}_l$, $\hat{x}_h$ denote the solutions to:*

$$\int_0^{\hat{x}_l} g_l(1-H(x))dx = c - f_h$$

and

$$\frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1-H(x))dx = \frac{1}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx.$$

respectively. The optimal menu of securities is given by $\hat{t}_l = \frac{c-f_h}{f_l}$, $\hat{t}_h = 1$, and

$$\hat{R}_l(x) = \begin{cases} \frac{x}{f_l}, \text{ for } x \leq \hat{x}_l \\ \frac{\hat{x}_l}{f_l}, \text{ otherwise} \end{cases}$$

and

$$\hat{R}_h(x) = \begin{cases} 0, \text{ for } x \leq \hat{x}_l \\ \frac{x-\hat{x}_l}{f_h}, \text{ for } \hat{x}_l \leq x \leq \hat{x}_h \\ \frac{\hat{x}_h-\hat{x}_l}{f_h}, \text{ otherwise} \end{cases}$$

The above Theorem establishes that the optimal mechanism is still a menu of two contracts: one of them senior debt with interest rate $\frac{\hat{x}_l}{c-f_h} - 1$, and the other one junior debt with interest rate $\frac{\hat{x}_h-\hat{x}_l}{f_h} - 1$. Relative to the benchmark model with a purchasing limit, the new menu offers the same interest rate for senior debt holders, but a higher interest rate for the junior debt holders. This happens because the more aggressive investors can now earn more by deviating and buying $\frac{1}{t_l}$ units of senior debt. Thus, in order to ensure incentive compatibility, the interest rate offered for the junior debt must increase.

Remark: It is intuitive that, in the present framework where investors are not constrained in their purchases, the interest rate for junior debt should be higher than the one for senior debt. To prove it, recall that the (IC-h) constraint now reads:

$$\int_X \left[R_h'(x) - R_l'(x)\right] g_h(1 - H(x))dx = 0$$
$$\Leftrightarrow \frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1 - H(x))dx = \frac{1}{c - f_h}\int_0^{\hat{x}_l} g_h(1 - H(x))dx$$

As the function $g_h(1 - H(x))$ is non-negative and decreasing, the following chain of inequalities immediately follows from (IC-h):

$$\begin{aligned}\frac{g_h(1 - H(\hat{x}_l))}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} dx &> \frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1 - H(x))dx \\ &= \frac{1}{c - f_h}\int_0^{\hat{x}_l} g_h(1 - H(x))dx \\ &> \frac{g_h(1 - H(\hat{x}_l))}{c - f_h}\int_0^{\hat{x}_l} dx\end{aligned}$$

The above inequalities further imply that

$$\frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} dx > \frac{1}{c - f_h}\int_0^{\hat{x}_l} dx \Rightarrow \frac{\hat{x}_h - \hat{x}_l}{f_h} - 1 > \frac{\hat{x}_l}{c - f_h} - 1$$

as desired.

Remark: The above result offers many intuitive implications that can be empirically tested. For example, in the aftermath of the financial crisis of 2007-2008, it has been argued that investors became more risk-averse, with possible negative consequences on the viability of tranched CDO's (collateralized debt obligations).[21] Our model makes clear predictions about how the optimal securities change in such a case.

Let us assume, for example, that the conservative investors become even more risk-averse, and thus use a more convex distortion $\overline{g}_l$. It directly follows that $\overline{g}_l(p) \le g_l(p)$ for any $p \in [0, 1]$. To meet the (IR-l) constraint, the new relevant cutoff $\overline{x}_l$ that solves

$$\int_0^{\overline{x}_l} g_l(1 - H(x))dx = c - f_h,$$

must exceed $\hat{x}_l$. This implies the interest rate, given by $\frac{\overline{x}_l}{c - f_h} - 1$, offered to these (more) conservative investors increases. This change also has implications for the optimal security $\overline{R}_h^*$ presented to aggressive investors, whose characteristics have not

[21] "What happened as a result of the crisis was that investors started asking more for a AAA tranche over similarly rated corporate bonds. Risk aversion increased, causing CDOs to fail." Jon Gregory, Solum Financials, cited in Sherif, N. (2014). The risk of risk aversion. *Risk*, **69**.

changed. Given (IC-h)

$$\int_{\overline{x}_l}^{\overline{x}_h} g_h(1-H(x))dx = \frac{f_h}{c-f_h}\int_0^{\overline{x}_l} g_h(1-H(x))dx$$

remains binding, the relevant new cutoff $\overline{x}_h$, and thus also the interest rates offered to the aggressive investors, must increase. The rise in interest rates, without a corresponding shift in the asset's return distribution, indicates an increased probability of failing to fulfill the obligations of the outstanding debt contracts. This finding aligns with the observations by Ospina and Uhlig [2018], who pointed out that mortgage-backed securities issued around the 2007-2008 financial crisis — a time when investors had become notably more risk-averse — were in fact more susceptible to defaults compared to those from earlier periods.

8 Conclusion

We have analyzed a novel security design model where an issuer raises capital from a population of heterogeneous, risk-averse, and budget-constrained investors. The issuer sells securities backed by an underlying asset with stochastic returns. Investors assess risk according to non-expected utility preferences that mirror the important class of *spectral* risk measures, including average value at risk, expected shortfall, and the family of exponential distortions, among others. Investors differ in their risk appetites and in their budgets, both of which are their private information. All agents (issuers and investors) possess the same information about the distribution of the asset's returns.

In this environment, we used the tools of mechanism design in order to derive the optimal security design. We found that the optimal mechanism partitions the asset's realized cash flow into several securities conforming to the commonly observed practice of tranching, where senior claims are paid before the subordinate ones. We explicitly derived the structure of the optimal menu of offered securities such as senior debt, junior debt and equity, and described how it depends on the model's main features such as the asset's volatility, the financing need, and the investors' degrees of risk aversion.

Appendix

Proof of Lemma 1. Suppose that there exists an optimal menu $(R_\theta(x), t_\theta)_{\theta\in\Theta}$ for which $f_h t_h + f_l t_l > c$ and $t_l > 0$. This implies that $R_l(x)$ is not equal to zero on a set of strictly positive measure. Then, we can construct another menu $(\tilde{R}_\theta(x), \tilde{t}_\theta)_{\theta\in\Theta}$ such that $\tilde{R}_h(x) = R_h(x)$ for all x and $\tilde{R}_l(x) = (1-\varepsilon)R_l(x)$ for all x and for some $\varepsilon > 0$.

Let $\tilde{t}_h = t_h$, and

$$\tilde{t}_l = t_l - \varepsilon \int_X R_l'(x) g_l (1 - H(x)) dx$$

There clearly exists a sufficiently small ε such that $f_h \tilde{t}_h + f_l \tilde{t}_l \geq c$. It is then easy to verify that the newly constructed mechanism is implementable as long as the original mechanism is implementable. Moreover, as

$$f_l(t_l - \tilde{t}_l) = f_l \varepsilon \int_X R_l'(x) g_l (1 - H(x)) dx < f_l \varepsilon \int_X R_l'(x)(1 - H(x)) dx,$$

we can conclude that the new mechanism is strictly more profitable than the original one. Thus, the original mechanism could not have been optimal, yielding a contradiction. The case where $t_l = 0$ can be proved in a similar way - we omit here the details. ■

Proof of Proposition 1. Let

$$V(R_h) = \int_X R_h'(x) g_h (1 - H(x)) dx$$

denote an aggressive investor's utility from holding security R_h, and let

$$C(R_h) = \int_X R_h'(x)(1 - H(x)) dx$$

denote the cost to the issuer of providing such a security (in terms of foregone cash-flow from the asset). Suppose that the debt contract R_h^* defined in the statement of the Proposition 1 is not optimal. Then, there exists another feasible mechanism $(\tilde{R}_h, t_h)$ such that $V(\tilde{R}_h) = V(R_h^*) = \frac{c}{f_h}$ so that (IR-h) binds, and such that $C(\tilde{R}_h) < C(R_h^*)$. It is clear that $\tilde{R}_h$ cannot be another debt contract. Then, by Theorem 1 there exists a debt contract R_h^D with cutoff x^{**} such that R_h^D second-order stochastically dominates any security R_h having the same provision cost $C(R_h) = C(\tilde{R}_h)$. Since the investor is risk-averse, we obtain that

$$V(R_h^D) \geq V(\tilde{R}_h) = V(R_h^*) = \frac{c}{f_h}.$$

where the equality follow from the construction of $\tilde{R}_h$). The above inequality, together with the observation that both R_h^* and R_h^D are debt contracts, imply that the interest rate offered in R_h^D must be higher than the one offered by R_h^*, so that $x^{**} \geq x^*$. This also implies that the cost of provision is higher $C(R_h^D) \geq C(R_h^*) > C(\tilde{R}_h)$, yielding a contradiction to the construction of $\tilde{R}_h$. ■

Proof of Lemma 2. Suppose that there exists a solution to the relaxed problem such that $t_l > 0$ and $t_h < 1$. Then, we can construct another menu $(\tilde{R}_\theta(x), \tilde{t}_\theta)_{\theta\in\Theta}$ that transfers a share ε, $0 < \varepsilon < 1$, of the asset designed for conservative investors to the asset designed to aggressive investors, i.e.

$$\tilde{R}_h(x) = R_h(x) + \frac{\varepsilon f_l}{f_h} R_l(x) \ , \ \tilde{R}_l(x) = (1-\varepsilon) R_l(x)$$

for all x. Further, let

$$\tilde{t}_h = t_h + \frac{\varepsilon f_l}{f_h} \int_X R_l'(x) g_h(1-H(x))dx \ , \ \tilde{t}_l = t_l - \varepsilon \int_X R_l'(x) g_l(1-H(x))dx.$$

For sufficiently small ε, $\tilde{t}_h < 1$ and $\tilde{t}_l > 0$. It is easy to verify that the new mechanism is implementable as long as the original mechanism is. Moreover, the change in the seller's profit is given by:

$$(f_h t_h + f_l t_l) - (f_h \tilde{t}_h + f_l \tilde{t}_l) = f_l \varepsilon \int_X R_l'(x) \left[g_h(1-H(x)) - g_l(1-H(x) \right] dx > 0.$$

Therefore, the original mechanism cannot be the solution to (Problem Q), yielding a contradiction. ■

In order to prove Theorem 2, we first prove the following Proposition.

Proposition 5 *Let x_l^*, x_h^* denote the solutions to:*

$$\int_0^{x_l^*} g_l(1-H(x))dx = c - f_h$$

and

$$\frac{1}{f_h} \int_{x_l^*}^{x_h^*} g_h(1-H(x))dx = \frac{1}{f_l} \int_0^{x_l^*} g_h(1-H(x))dx + \frac{1-c}{f_l}$$

respectively. The solution to (Problem Q') is given by:

$$R_l^*(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \leq x_l^* \\ \frac{x_l^*}{f_l}, \text{ otherwise} \end{cases}$$

$$R_h^*(x) = \begin{cases} 0, \text{ for } x \leq x_l^* \\ \frac{x - x_l^*}{f_h}, \text{ for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^* - x_l^*}{f_h}, \text{ otherwise} \end{cases}$$

Proof of Proposition 5. For notational convenience, let

$$\phi(x) = f_l R_l'(x) + f_h R_h'(x).$$

denote the *slope* of the offered aggregate securities, and observe that

$$\phi(x) - R_l'(x) = f_h \left[R_h'(x) - R_l'(x)\right] .$$

Intuitively, $\phi(x)$ is the share of an additional dollar of the project's return that is allocated to investors if the current return equals x. Then (Problem Q') can be rewritten as:

$$\begin{aligned}
\text{(Problem } Q''\text{)} \min_{\phi, R_l} & \left\{ \int_X \phi(x)(1-H(x))dx \right\} \\
s.t. & \int_X R_l'(x) g_l (1-H(x))dx = \frac{c-f_h}{f_l} \\
& \int_X [\phi(x) - R_l'(x)] g_h (1-H(x))dx = \frac{(1-c)f_h}{f_l} \\
& \phi(x) \geq f_l R_l'(x) \geq 0, \ \forall x \in X \\
& \int_0^x \phi(z)dz \leq x, \ \forall x \in X \\
& R_l(0) = 0
\end{aligned}$$

We further relax the above problem by ignoring the third constraint, i.e. that says that the share given to both investor types must exceed the share given to conservative investors. The problem then becomes:

$$\begin{aligned}
\text{(Problem } Q'''\text{)} \min_{\phi, R_l} & \left\{ \int_X \phi(x)(1-H(x))dx \right\} \\
s.t. & \int_X R_l'(x) g_l (1-H(x))dx = \frac{c-f_h}{f_l} \\
& \int_X [\phi(x) - R_l'(x)] g_h (1-H(x))dx = \frac{(1-c)f_h}{f_l} \\
& \int_0^x \phi(z)dz \leq x, \ \forall x \in X \\
& \phi(x), R_l'(x) \geq 0, \ \forall x \in X \\
& R_l(0) = 0
\end{aligned}$$

If the solution to the above problem satisfies all the constraints in (Problem Q''), then it is also a solution to (Problem Q''). For solving (Problem Q''') we proceed in two steps:

Step 1: We first keep the function R_l' fixed. Then we need to solve:

$$\begin{aligned}
&\min_{\phi}\left\{\int_X \phi(x)(1-H(x))dx\right\}\\
s.t. &\int_X \phi(x)g_h(1-H(x))dx = \int_X R_l'(x)g_h(1-H(x))dx + \frac{(1-c)f_h}{f_l}\\
&\int_0^x \phi(z)dz \le x,\ \forall x \in X\\
&\phi(x) \ge 0, \forall x \in X
\end{aligned}$$

Since g_h is convex, the same argument as in the proof for Proposition 1 yields that the optimal average security corresponds to a debt contract, and is given by $\phi(x) = \mathbf{1}_{x \le x_h^*}$ where x_h^* solves

$$\int_0^{x_h^*} g_h(1-H(x))dx = \int_X R_l'(x)g_h(1-H(x))dx + \frac{(1-c)f_h}{f_l}.$$

Step 2: Next, the seller must optimally chooses the function R_l in order to minimize x_h^* (i.e., relax as much as possible the isoperimetric constraint) while satisfying all the other remaining constraints. Minimizing x_h^* is equivalent to the minimization problem

$$\min_{R_l}\left\{\int_X R_l'(x)g_h(1-H(x))dx + \frac{(1-c)f_h}{f_l}\right\}$$

under the same constraints. Since $\frac{(1-c)f_h}{f_l}$ is a constant, the issuer's problem in this second step reduces to:

$$\begin{aligned}
&\min_{R_l}\left\{\int_X R_l'(x)g_h(1-H(x))dx\right\}\\
s.t. &\int_X R_l'(x)g_l(1-H(x))dx = \frac{c-f_h}{f_l}\\
&f_l\int_0^x R_l'(z)dz \le x\\
&R_l'(x) \ge 0,\ \forall x \in X\\
&R_l(0) = 0
\end{aligned}$$

Recall that g_l is a convex transformation of g_h. By assumption, there exists an increasing and convex function k such that $g_l(z) = k(g_h(z))$. It must be the case that $k(0) = 0$ and $k(1) = 1$.

Consider a new, artificial asset whose return is governed by the distribution $\tilde{H}$:

$X \to [0,1]$ defined by

$$1 - \tilde{H}(x) = g_h(1 - H(x)) \text{ for all } x \in X$$

The above problem can be then rewritten as:

$$\begin{aligned}
&\min_{R_l} \left\{ \int_X R_l'(x)(1 - \tilde{H}(x))dx \right\} \\
&s.t. \int_X R_l'(x)k(1 - \tilde{H}(x))dx = \frac{c - f_h}{f_l} \\
&f_l \int_0^x R_l'(z)dz \leq x, \ \forall x \in X \\
&R_l'(x) \geq 0, \ \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

We can then apply the same argument as in Step 1, and obtain that the solution to the above problem is $(R_l^*)'(x) = \frac{1}{f_l}\mathbf{1}_{x \leq x_l^*}$ where x_l^* solves

$$\frac{1}{f_l} \int_0^{x_l^*} g_l(1 - H(x))dx = \frac{c - f_h}{f_l}.$$

It directly follows that

$$\phi^*(x) = \frac{1}{f_l}\mathbf{1}_{x \leq x_h^*}$$

where x_h^* solves

$$\begin{aligned}
\int_0^{x_h^*} g_h(1 - H(x))dx =& \int_X (R_l^*)'(x) g_h(1 - H(x))dx + \frac{(1-c)f_h}{f_l} \\
=& \frac{1}{f_l} \int_0^{x_l^*} g_h(1 - H(x))dx + \frac{(1-c)f_h}{f_l},
\end{aligned} \tag{1}$$

which is eqiuvalent to

$$\frac{1}{f_h} \int_{x_l^*}^{x_h^*} g_h(1 - H(x))dx = \frac{1}{f_l} \int_0^{x_l^*} g_h(1 - H(x))dx + \frac{1-c}{f_l}$$

as desired. In addition, we obtain.

$$(R_h^*)'(x) = \frac{1}{f_h}(\phi(x) - f_l R_l'(x)) = \frac{1}{f_h}\mathbf{1}_{x_l^* \leq x \leq x_h^*}.$$

To conclude, we have obtained that the optimal menu of securities for (Problem

Q''') is given by:

$$R_l^*(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \leq x_l^* \\ \frac{x_l^*}{f_l}, \text{ otherwise} \end{cases}$$

$$R_h^*(x) = \begin{cases} 0, \text{ for } x \leq x_l^* \\ \frac{x-x_l^*}{f_h}, \text{ for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^*-x_l^*}{f_h}, \text{ otherwise} \end{cases}$$

In order to verify that the above menu is also a solution to (Problem Q'), we still need to check that the ignored constraint,

$$\phi(x) \geq f_l R_l'(x) \geq 0$$

also holds. For the last inequality to hold, it suffices to verify that $x_l^* \leq x_h^*$, and this follows from Equation (1):

$$\begin{aligned} \int_0^{x_h^*} g_h(1 - H(x))dx =& \frac{1}{f_l} \int_0^{x_l^*} g_h(1 - H(x))dx + \frac{(1-c)f_h}{f_l} \\ \geq& \int_0^{x_l^*} g_l(1 - H(x))dx. \end{aligned}$$

QED ■

Proof of Theorem 2. In order to prove that R_l^* and R_h^* as described in Proposition 5 are the optimal securities, we still need to show that the omitted constraints, namely (IR-h) and (IC-l), are also satisfied. The fact that (IR-h) holds follows directly from (IR-l) and (IC-h). (IC-l) requires that

$$\begin{aligned} \int_X \left[R_h'(x) - R_l'(x)\right] g_l(1 - H(x))dx \leq& t_h - t_l = \frac{1-c}{f_l} \\ \Leftrightarrow \int_X \left[R_h'(x) - R_l'(x)\right] \left[g_h(1 - H(x)) - g_l(1 - H(x))\right]dx \geq 0& \end{aligned}$$

The equivalence holds because

$$\int_X \left[R_h'(x) - R_l'(x)\right] g_h(1 - H(x))dx = \frac{1-c}{f_l}$$

by the (IC-h) constraint. In order to prove (IC-l), we proceed as follows:

For any fixed $y \in X$, consider two distributions defined on the interval $[0, y]$ by

$$\pi_{yh}(x) = \frac{\int_0^x g_h(1 - H(z))dz}{\int_0^y g_h(1 - H(z))dz}; \quad \pi_{yl}(x) = \frac{\int_0^x g_l(1 - H(z))dz}{\int_0^y g_l(1 - H(z))dz}$$

The ratio of the respective densities is given by

$$\frac{\pi'_{yh}(x)}{\pi'_{yl}(x)} = \frac{\int_0^y g_l(1-H(z))dz}{\int_0^y g_h(1-H(z))dz} \cdot \frac{g_h(1-H(x))}{g_l(1-H(x))}$$

This ratio is increasing in x because we assumed that g_l is a convex transformation of g_h, which implies that $\frac{g_h(x)}{g_l(x)}$ is decreasing. This further implies that $\pi_{yh}(x) \succeq_{LR} \pi_{yl}(x)$ where LR denotes the likelihood ratio stochastic order. It is well-known (see Shaked and Shanthikumar [2007], Theorems 1.B.1, page 18 and 1.C.1, page 43) that the likelihood ratio stochastic order implies the hazard rate order, and that the latter implies the usual first order stochastic dominance. Hence we obtain that $\pi_{yh}(x) \succeq_{FOSD} \pi_{yl}(x)$ for each $y \in X$.

Note that

$$R'_h(x) - R'_l(x) = \frac{1}{f_h}\mathbf{1}_{x_l^* \leq x \leq x_h^*} - \frac{1}{f_l}\mathbf{1}_{x \leq x_l^*} = \begin{cases} -\frac{1}{f_l},\ x \leq x_l^* \\ \frac{1}{f_h},\ x_l^* \leq x \leq x_h^* \\ 0,\ x \geq x_h^* \end{cases}$$

is an increasing function on $[0, x_h^*]$. Applying the above observation about stochastic dominance to $y = x_h^*$, and recalling that the expectation of an increasing function increases under a FOSD shift, we obtain that:

$$\frac{\int_0^{x_h^*}[R'_h(x) - R'_l(x)]g_h(1-H(x))dx}{\int_0^{x_h^*} g_h(1-H(z))dz} \geq \frac{\int_0^{x_h^*}[R_h(x) - R'_l(x)]g_l(1-H(x))dx}{\int_0^{x_h^*} g_l(1-H(z))dz}$$

As $g_h(1-H(x)) \geq g_l(1-H(x))$ for all x we have

$$\frac{1}{\int_0^{x_h^*} g_h(1-H(z))dz} \leq \frac{1}{\int_0^{x_h^*} g_l(1-H(z))dz}$$

Together with $R'_h(x) - R'_l(x) = 0$ for $x \geq x_h^*$, the two inequalities above together imply that

$$\int_X [R'_h(x) - R'_l(x)]\, g_h(1-H(x))dx \geq \int_X [R'_h(x) - R'_l(x)]\, g_l(1-H(x))dx$$

as desired. ■

In order to prove Proposition 2, we first need a Lemma. Consider two assets, x and y, with distributions of returns H_x and H_y, respectively, such that either x FOSD y or x SOSD y.

Lemma 3 *Let $\hat{x}$ denote the solution to*

$$\int_0^{\hat{x}} (1 - H_x(z))dz = \int_0^{y^*} (1 - H_y(z))dz$$

Then

$$\int_0^{\hat{x}} g_h(1 - H_x(z))dz \geq \int_0^{y^*} g_h(1 - H_y(z))dz.$$

Proof of Lemma 3. Define

$$H_x^D(t) = \begin{cases} H_x(t) \text{ for } x \leq \hat{x} \\ 1, \text{ otherwise} \end{cases} \quad \text{and } H_y^D(t) = \begin{cases} H_y(t) \text{ for } t \leq y^* \\ 1, \text{ otherwise} \end{cases}$$

H_x^D (H_y^D) describes the distribution of a debt contract that is backed by asset $x(y)$ with cutoff $\hat{x}$ (y^*). The expected values of these two debt contracts are, by construction, the same:

$$\begin{aligned}
\int_0^{\overline{x}} (1 - H_x^D(z))dz &= \int_0^{\hat{x}} (1 - H_x^D(z))dz + \int_{\hat{x}}^{\overline{x}} (1 - H_x^D(z))dz \\
&= \int_0^{\hat{x}} (1 - H_x(z))dz + \int_{\hat{x}}^{\overline{x}} (1 - 1)dz \\
&= \int_0^{y^*} (1 - H_y(z))dz = \int_0^{\overline{x}} (1 - H_y^D(z))dz
\end{aligned}$$

By the assumption that asset x SOSD asset y, and by the definition of $\hat{x}$, we know that $\hat{x} \leq y^*$. Further, for any $s \in (\hat{x}, y^*)$ it holds that:

$$\int_0^s (1 - H_x^D(z))dz = \int_0^{\hat{x}} (1 - H_x^D(z))dz = \int_0^{y^*} (1 - H_y^D(z))dz > \int_0^s (1 - H_y^D(z))dz$$

For any $s < \hat{x}$ it holds that:

$$\int_0^s (1 - H_x^D(z))dz = \int_0^s (1 - H_x(z))dz \geq \int_0^s (1 - H_y(z))dz = \int_0^s (1 - H_y^D(z))dz$$

which directly follow from the assumption that x SOSD y.

We can then conclude that H_x^D SOSD H_y^D. As investors are risk averse, we obtain

$$\int_0^{\hat{x}} g_h(1 - H_x(z))dz = \int_0^{\hat{x}} g_h(1 - H_x^D(z))dz \geq \int_0^{y^*} g_h(1 - H_y^D(z))dz.$$

as desired. ■

Proof of Proposition 2. We give the proof for SOSD. The proof for FOSD is similar and we omit the details.

Consider two assets, x and y, with distributions of returns H_x and H_y, respectively, such that either x FOSD y or x SOSD y. Letting x_l^*, y_l^*, x_h^* and y_h^* denote the solutions to:

$$\text{(IR-l)} \int_0^{x_l^*} g_l(1 - H_x(t))dt = \int_0^{y_l^*} g_l(1 - H_y(t))dt = c - f_h$$

and

$$\begin{aligned}\text{(IC-h)} \ & \frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1 - H_x(t))dt - \frac{1}{f_l}\int_0^{x_l^*} g_h(1 - H_x(t))dt \\ =& \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1 - H_x(t))dt - \frac{1}{f_l}\int_0^{y_l^*} g_h(1 - H_x(t))dt = 1 - \frac{c - f_h}{f_l}\end{aligned}$$

Then, $x_l^* \leq y_l^*$ directly follows from the definition of SOSD and from the agents' risk aversion. The rest of proof consists of 3 steps.

Step 1: Let $\tilde{x}$ denote the solution to

$$\int_0^{\tilde{x}}[1 - H_x(t)]dt = \int_0^{y_h^*}[1 - H_y(t)]dt$$

It follows from Lemma 3 that:

$$\int_0^{\tilde{x}} g_h(1 - H_x(t))dt \geq \int_0^{y_h^*} g_h(1 - H_y(t))dt$$

Step 2: We next show that

$$\int_0^{x_l^*} g_h(1 - H_x(t))dt \leq \int_0^{y_l^*} g_h(1 - H_y(t))dt$$

This follows from

$$\int_0^{x_l^*} g_l(1 - H_x(t))dt = \int_0^{y_l^*} g_l(1 - H_y(t))dt$$

and from the assumption that g_l is more convex than g_h (i.e., because type l, the conservative type, is more risk-averse than type h, the aggressive type) .

Step 3: Steps 1 and 2 imply together that

$$\int_{x_l^*}^{\tilde{x}} g_h(1 - H_x(t))dt \geq \int_{y_l^*}^{y_h^*} g_h(1 - H_y(t))dt$$

which further implies

$$\frac{1}{f_h}\int_{x_l^*}^{\tilde{x}} g_h(1-H_x(t))dt - \frac{1}{f_l}\int_0^{x_l^*} g_h(1-H_x(t))dt$$
$$\geq \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1-H_y(t))dt - \frac{1}{f_l}\int_0^{y_l^*} g_h(1-H_y(t))dt$$

Recall that x_h^* and y_h^* solve

$$\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1-H_x(t))dt - \frac{1}{f_l}\int_0^{x_l^*} g_h(1-H_x(t))dt$$
$$= \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1-H_y(t))dt - \frac{1}{f_l}\int_0^{y_l^*} g_h(1-H_y(t))dt$$

It follows that $x_h^* \leq \tilde{x}$, and thus that

$$\int_0^{x_h^*}(1-H_x(z))dz \leq \int_0^{\tilde{x}}(1-H_x(z))dz = \int_0^{y_h^*}(1-H_y(z))dz$$

which proves Part 2 of Proposition 2. It also follows that

$$\begin{aligned}\frac{1}{f_h}\int_{x_l^*}^{x_h^*} g_h(1-H_x(z))dz &= \frac{1}{f_l}\int_0^{x_l^*} g_h(1-H_x(z))dz + (1-\frac{c-f_h}{f_l})\\ &\leq \frac{1}{f_l}\int_0^{y_l^*} g_h(1-H_y(z))dz + (1-\frac{c-f_h}{f_l})\\ &= \frac{1}{f_h}\int_{y_l^*}^{y_h^*} g_h(1-H_y(z))dz,\end{aligned}$$

which proves Part 3 of Proposition 2.

The only remaining task is to show that under a FOSD risk $x_h^* - x_l^* \leq y_h^* - y_l^*$. Suppose not: then

$$x_h^* - x_l^* > y_h^* - y_l^* \equiv \Delta$$

Since $x_l^* < y_l^*$ and g_h is non-decreasing, we obtain:

$$\begin{aligned}\int_{x_l^*}^{x_h^*} g_h(1-H_x(z))dz &> \int_{x_l^*}^{x_l^*+\Delta} g_h(1-H_x(z))dz\\ &\geq \int_{y_l^*}^{y_l^*+\Delta} g_h(1-H_x(z))dz \geq \int_{y_l^*}^{y_h^*} g_h(1-H_y(z))dz\end{aligned}$$

where the last inequality follows from FOSD. This leads to a contradiction since we proved above that

$$\int_{x_l^*}^{x_h^*} g_h(1-H_x(z))dz \geq \int_{y_l^*}^{y_h^*} g_h(1-H_y(z))dz.$$

We therefore conclude that $x_h^* - x_l^* \leq y_h^* - y_l^*$ if x FOSD y. ■

Proof of Proposition 3. Take any c, let $x_l^*(c)$ denote the cutoff point of the senior debt, which solves (IR-l):

$$\int_0^{x_l(c)} g_l(1-H(z))dz = c - f_h\,.$$

Taking the derivative with respect to c twice in the above expression yields:

$$\begin{aligned} g_l(1-H(x_l(c)))\cdot x_l'(c) &= 1 \\ -g_l'(1-H(x_l(c))h(x_l(c))\cdot[x_l'(c)]^2 + g_l(1-H(x_l(c)))\cdot x_l''(c) &= 0 \end{aligned}$$

The second equation yields $x_l''(c) \geq 0$. Moreover, clearly $x_l(0) = 0$. Hence the corresponding interest rate, $\frac{x_l(c)}{c-f_h} - 1$ is increasing in the financing need c.

Similarly, $x_h(c)$, which denotes the cutoff point for the junior debt, is given by (IC-h):

$$\begin{aligned} &\frac{1}{f_h}\int_{x_l(c)}^{x_h(c)} g_h(1-H(t))dt - \frac{1}{f_l}\int_0^{x_l(c)} g_h(1-H(t))dt = 1 - \frac{c-f_h}{f_l} \\ \Rightarrow &\frac{1}{f_h}\int_{x_l(c)}^{x_h(c)} g_h(1-H(t))dt = 1 + \frac{1}{f_l}\int_0^{x_l(c)} [g_h(1-H(t)) - g_l(1-H(t))]dt \end{aligned}$$

As c increase, $x_l(c)$ increases. Moreover, $g_h(1-H(t)) - g_l(1-H(t)) \geq 0$. As the right hand side of the equation increases, the left hand side of the equation, $\int_{x_l(c)}^{x_h(c)} g_h(1 - H_x(z))dz$ must increase as well.

Finally, we want to prove that $x_h(c) - x_l(c)$ also increases as c increases. Suppose that this is not the case. Then there must exist $c_1 > c_2$ such that $x_h(c_1) - x_l(c_1) < x_h(c_2) - x_l(c_2)$. Let $\Delta = x_h(c_2) - x_l(c_2)$. Since $g_h(1 - H(t))$ is decreasing in t and because $x_l(c_1) > x_l(c_2)$, we have

$$\begin{aligned} \int_{x_l(c_1)}^{x_h(c_1)} g_h(1-H_x(z))dz &< \int_{x_l(c_1)}^{x_l(c_1)+\Delta} g_h(1-H_x(z))dz \\ &\leq \int_{x_l(c_2)}^{x_l(c_2)+\Delta} g_h(1-H_x(z))dz = \int_{x_l(c_2)}^{x_h(c_2)} g_h(1-H_x(z))dz \end{aligned}$$

which contradicts the result obtained above. Therefore, it must hold that $x_h(c) - x_l(c)$

increases as c increases. ■

Proof of Theorem 3. The proof consists of three main steps.

Step 1: Suppose that the agents' budget types are public information while the risk types remain the agents' private information, as before. We show that there exists an optimal menu such that $R^*_{l,\beta}(x) = \beta R^*_{l,1}(x)$, $R^*_{h,\beta}(x) = \beta R^*_{h,1}(x)$ for all x, $t^*_{l,\beta} = \beta t^*_{l,1}$, and $t^*_{h,\beta} = \beta t^*_{h,1}$.

Step 1-a: We first show that if there exists an optimal mechanism $(R^*_{\theta b}, t^*_{\theta b})$ for which $R^*_{l,\beta}(x) \neq \beta R^*_{l,1}(x)$, then we can construct another optimal mechanism $(\tilde{R}^*_{\theta b}, \tilde{t}^*_{\theta b})$ such that $\tilde{R}^*_{l,\beta}(x) = \beta \tilde{R}^*_{l,1}(x)$.

Observe that in any optimal mechanism, the constraints (IR-$l1$), (IR-$l\beta$), (IC-$h1$), and (IC-$h\beta$) must all bind:

$$
\begin{aligned}
(IR - l1): &\int_X R'_{l,1}(x) g_l (1 - H(x)) dx = t_{l,1} \\
(IR - l\beta): &\int_X R'_{l,\beta}(x) g_l (1 - H(x)) dx = t_{l,\beta} \\
(IC - h1): &\int_X [R'_{h,1}(x) - R'_{l,1}(x)] g_h (1 - H(x)) dx = t_{h,1} - t_{l,1} \\
(IC - h\beta): &\int_X [R'_{h,\beta}(x) - R'_{l,\beta}(x)] g_h (1 - H(x)) dx = t_{h,\beta} - t_{l,\beta}
\end{aligned}
$$

Putting the above equations together yields:

$$
\begin{aligned}
&p \int_X [R'_{h,\beta}(x) - R'_{l,\beta}(x)] g_h (1 - H(x)) dx + (1-p) \int_X [R'_{h,1}(x) - R'_{l,1}(x] g_h (1 - H(x)) dx \\
&= p t_{h,\beta} + (1-p) t_{h,1} - \int_X [p R'_{l,\beta}(x) + (1-p) R'_{l,1}(x)] g_l (1 - H(x)) dx \\
\Rightarrow &\int_X [p R'_{h,\beta}(x) + (1-p) R'_{h,1}(x)] g_h (1 - H(x)) dx - p t_{h,\beta} - (1-p) t_{h,1} \\
&= \int_X [p R'_{l,1}(x) + (1-p) R'_{l,\beta}(x)] [g_h (1 - H(x)) - g_l (1 - H(x)] dx
\end{aligned}
$$

Thus, as long as the total asset assigned to conservative investors remains unchanged, i.e. as long as

$$
p R^*_{l,\beta}(x) + (1-p) R^*_{l,1}(x) = p \tilde{R}^*_{l,\beta}(x) + (1-p) \tilde{R}^*_{l,1}(x) \ \forall x,
$$

we can construct another incentive compatible mechanism where the total asset assigned to the aggressive investors and their total expected payment are also unchanged. If the original mechanism was optimal, so is the new one.

Step 1-b: By (IR-$l1$) and (IR-$l\beta$), in the newly constructed mechanism $(\tilde{R}^*_{\theta b}, \tilde{t}^*_{\theta b})$, $\tilde{R}^*_{l,\beta}(x) = \beta \tilde{R}^*_{l,1}(x)$ implies $\tilde{t}^*_{l\beta} = \beta \tilde{t}^*_{l1}$.

Step 1-c: Suppose now that $\tilde{t}^*_{h,\beta} \neq \beta\tilde{t}^*_{h,1}$. This means that the budget of aggressive investors are not exhausted. By using similar arguments to those in Lemma 2, it can be verified that such a mechanism cannot be optimal.

Finally, steps (1.a)-(1.c) together imply that $\tilde{R}^*_{h,\beta}(x) = \beta\tilde{R}^*_{h,1}(x)$.

Step 2: By Step 1, assuming that the agents' budget types are public information, we can restrict attention to the class of menus that satisfy $R^*_{l,\beta}(x) = \beta R^*_{l,1}(x)$, $R^*_{h,\beta}(x) = \beta R^*_{h,1}(x)$ for all x, $t^*_{l,\beta} = \beta t^*_{l,1}$, and $t^*_{h,\beta} = \beta t^*_{h,1}$. Then by following essentially the same arguments as in the proof of Theorem 2, we can show that the mechanism described in Theorem 3 is optimal in this class.

Step 3: The remaining step is to verify that, even when budget types are private information, the mechanism described in Theorem 3 is implementable, and thus optimal. It is clear that the individual rationality constraints for all types remain the same, so that they are satisfied. Moreover, as in the public budget setting, no agent has incentive to pretend to be another agent with the same budget type but different risk type. We show below that either an agent has no incentive to pretend to be another agent with the same risk type but different budget type, or he is unable to do so:

a Type $l1$ has no incentive to pretend to be of type $l\beta$ since in either case he will earn a payoff of 0 (this follows from the homogeneity of dual utility).

b Type $l\beta$ may not have enough money ($\beta < t_{l1}$) to pretend to be type $l1$. Even if $\beta > t_{l1}$, type $l\beta$ still has no incentive to pretend to be of type $l1$ since in either case he will earn a payoff of 0.

c Type $h\beta$ cannot pretend to be type $h1$ since he does not have enough money to do so ($\beta < 1 = t_{h1}$).

d Finally, type $h1$ has no incentive to pretend to be of type $h\beta$ since:

$$\begin{aligned}\int_X R'_{h,1}(x)g_h(1-H(x))dx - t_{h,1} =& \frac{1}{\beta}[\int_X R'_{h,\beta}(x)g_h(1-H(x))dx - t_{h,\beta}] \\ >& \int_X R'_{h,\beta}(x)g_h(1-H(x))dx - t_{h,\beta}\end{aligned}$$

Finally, no type of investor wants here to misreport in both dimensions: since an agent who misreports his budget essentially "adopts" the utility function of that budget type, the observation follows from the standard incentive compatibility constraint with respect to deviations in the risk type only. To conclude, even if budget types are private information, the mechanism described in Theorem 3 is implementable, and yields the same expected profit as in the case with public budget. Therefore, it must be an optimal mechanism. ■

Proof of Proposition 4. By assumption, there exists an increasing and convex function $k(\cdot)$ such that $g(z) = k(g_h(z))$. It must be the case that $k(0) = 0$ and $k(1) = 1$.

As in the benchmark model, in order to solve the security design problem, we first derive the optimal mechanism for the relaxed problem where we impose neither (IC-l) nor (IR-h). We later check that the obtained solution for the relaxed problem indeed satisfies these omitted constraints. Formally, the relaxed problem is:

$$\begin{aligned}
&\min_{R_l,R_h} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)] g(1-H(x))dx \right\} \\
&\text{s.t.} \int_X R_l'(x) g_l(1-H(x))dx = \frac{c-f_h}{f_l} \\
&\quad \int_X [R_h'(x) - R_l'(x)] g_h(1-H(x))dx = t_h - t_l = \frac{1-c}{f_l} \\
&\quad R_h'(x), R_l'(x) \geq 0 \ \forall x \in X \\
&\quad f_l R_l'(x) + f_h R_h'(x) \leq 1 \ \forall x \in X \\
&\quad R_h(0), R_l(0) = 0
\end{aligned}$$

The proof follows a similar procedure to that of Proposition 5. We first fix R_l, and look at the following relaxed problem:

$$\begin{aligned}
&\min_{R_h} \left\{ \int_X R_h'(x) g(1-H(x))dx \right\} \\
&s.t. \int_X R_h'(x) g_h(1-H(x))dx = \int_X R_l'(x) g_h(1-H(x))dx + \frac{1-c}{f_l} \\
&0 \leq R_h'(x) \leq \frac{1}{f_h}, \ \forall x \in X \\
&R_h(0) = 0
\end{aligned}$$

Consider a new, artificial asset whose return is governed by the distribution $\tilde{H} : X \to [0,1]$ defined by

$$1 - \tilde{H}(x) = g_h(1-H(x)) \text{ for all } x \in X$$

Then, the above problem can be rewritten as follows:

$$\begin{aligned}
&\min_{R_h}\left\{\int_X R_h'(x)k(1-\tilde{H}(x))dx\right\}\\
s.t.\ &\int_X R_h'(x)(1-\tilde{H}(x))dx = \int_X R_l'(x)(1-\tilde{H}(x))dx + \frac{1-c}{f_l}\\
&0 \le R_h'(x) \le \frac{1}{f_h},\ \forall x \in X\\
&R_h(0) = 0
\end{aligned}$$

Let

$$\tilde{V}(R_h) = \int_X R_h'(x)k(1-\tilde{H}(x))dx;\ \tilde{C}(R_h) = \int_X R_h'(x)(1-\tilde{H}(x))dx$$

denote the utility derived from holding security R_h by an agent whose dual risk preference is described by the distortion k, and the cost to a risk-neutral seller of issuing such a security, respectively.

The issuer's problem is thus equivalent to the design of a doubly monotonic security that **minimizes** the agent's utility while keeping the expected cost fixed. Then, by Theorem 1, the optimal security has the form of an equity:

$$\tilde{R}_h(x) = \begin{cases} 0 \text{ for } x \le \tilde{x}_h \\ \frac{x-\tilde{x}_h}{f_h}, \text{ otherwise} \end{cases}$$

where $\tilde{x}_h$ is the solution to

$$\frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1-H(x))dx = \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l}$$

By following essentially the same procedure as in the proof of Proposition 5, we obtain that the optimal R_l takes the form of senior debt, and is given by:

$$\tilde{R}_l(x) = \begin{cases} \frac{x}{f_l} \text{ for } x \le \tilde{x}_l \\ \frac{\tilde{x}_l}{f_l}, \text{ otherwise} \end{cases}$$

where $\tilde{x}_l$ solves

$$\int_0^{\tilde{x}_l} g_l(1-H(x))dx = c - f_h$$

It follows that

$$\begin{aligned}
\frac{1}{f_h}\int_{\tilde{x}_h}^{\bar{x}} g_h(1-H(x))dx &= \int_X R_l'(x)g_h(1-H(x))dx + \frac{1-c}{f_l}\\
&= \frac{1}{f_l}\int_0^{\tilde{x}_l} g_h(1-H(x))dx + \frac{1-c}{f_l}
\end{aligned}$$

The last step is to check the menu described in Proposition 4 satisfies the ignored constraints (IR-h) and (IC-l). Note that

$$\tilde{R}_h'(x) - \tilde{R}_l'(x) = \begin{cases} -\frac{1}{f_l},\ x \leq \tilde{x}_l \\ 0,\ \tilde{x}_l \leq x \leq \tilde{x}_h \\ \frac{1}{f_h}\ x \geq \tilde{x}_h \end{cases}$$

increases on $[0, \overline{x}]$. Then, we can use the same arguments as that in the proof of Theorem 2 to show that the two ignored constraints are satisfied. ■

Proof of Theorem 4. We let

$$\phi(x) = f_h R_h'(x) + f_l R_l'(x)$$

be the slope of the offered aggregate securities as in the proof of Proposition 5. It follows that

$$\frac{1}{f_h}\left[\phi(x) - \frac{cf_l}{c - f_h} R_l'(x)\right] = R_h'(x) - \frac{f_l}{c - f_h} R_l'(x)$$

and the issuer's relaxed problem becomes:

$$\begin{aligned}
&\min_{R_h, R_l} \left\{ c \int_X \phi(x)(1 - H(x))dx \right\} \\
s.t. &\int_X R_l'(x) g_l (1 - H(x))dx = 1 \\
&\int_X \phi(x) g_h (1 - H(x))dx = \frac{f_l c}{c - f_h} \int_X R_l'(x) g_h (1 - H(x))dx \\
&\phi(x) \geq f_l R_l'(x) \geq 0, && \forall x \in X \\
&c \int_0^x \phi(z)dz \leq x, && \forall x \in X \\
&R_l(0) = 0
\end{aligned}$$

We can solve the above problem following a procedure that is similar to the one we used in Section 4.2. First, fixing the security R_l, the optimal avaerage slope ϕ is given by $\phi(x) = \mathbf{1}_{x \leq \hat{x}_h}$ where $\hat{x}_h$ is the solution to

$$\int_0^{\hat{x}_h} g_h (1 - H(x))dx = \frac{f_l c}{c - f_h} \int_X R_l'(x) g_h (1 - H(x))dx.$$

This yields a debt contract. Next, the seller must choose the optimal security R_l in order to minimize

$$\int_X \phi(x)(1 - H(x))dx$$

This is equivalent to

$$\begin{aligned}&\min_{R_l}\left\{\int_X R_l'(x)g_h(1-H(x))dx\right\}\\ s.t.&\int_X R_l'(x)g_l(1-H(x))dx=\frac{c-f_h}{f_l}\\ &\int_0^x R_l'(z)dz\le\frac{1}{f_l}\min\{x,\hat{x}_h\}\,,\forall x\in X\end{aligned}$$

Since g_l is a convex transformation of g_h, then, again by the same argument as in Section 4.2, we obtain that the optimal security R_l satisfies $R_l'(x)=\frac{1}{f_l}\mathbf{1}_{x\le\hat{x}_l}$ where $\hat{x}_l$ is the solution to the equation

$$\int_0^{\hat{x}_l} g_l(1-H(x))dx=c-f_h.$$

It can be then easily computed that $\hat{x}_h$ solves

$$\int_0^{\hat{x}_h} g_h(1-H(x))dx=\frac{c}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx.$$

which is equivalent to

$$\frac{1}{f_h}\int_{\hat{x}_l}^{\hat{x}_h} g_h(1-H(x))dx=\frac{1}{c-f_h}\int_0^{\hat{x}_l} g_h(1-H(x))dx$$

as desired. It follows that the optimal securities R_l and R_h are now given by:

$$R_l^*(x)=\begin{cases}\frac{x}{f_l},\text{ for } x\le\hat{x}_l\\ \frac{\hat{x}_l}{c-f_h},\text{ otherwise}\end{cases}$$

and

$$R_h^*(x)=\begin{cases}0,\text{ for } x\le\hat{x}_l\\ \frac{x-\hat{x}_l}{f_h},\text{ for } \hat{x}_l\le x\le\hat{x}_h\\ \frac{\hat{x}_h-\hat{x}_l}{f_h},\text{ otherwise}\end{cases}$$

The omitted (IC-l) constraint holds by the same argument as in the proof for Theorem 2. ■

Reinvestment is Never Strictly Optimal: As noted in Section 3.1, one can extend the model by allowing the issuer to use capital collected from investors in order to increase some payments to other investors. Then, the feasibility constraint becomes:

$$\sum_{\theta\in\{l,h\}} f_\theta\,R_\theta(x)\le x+\sum_\theta f_\theta t_\theta-c.$$

In oder to demonstrate that such a strategy is never strictly optimal, we consider two cases.

Case 1: Suppose that only aggressive investors participate in the optimal mechanism and that the issuer raises $c+\Delta$, where $\Delta > 0$. Then, the optimal issued security must be a solution to the following problem:

$$\begin{aligned}
&\min_{R_h} \left\{ f_h \int_X R_h'(x)(1-H(x))dx \right\} \\
\text{s.t.} &\int_X R_h'(x) g_h (1-H(x))dx = \frac{c+\Delta}{f_h} \\
&R_h'(x) \geq 0, && \forall x \in X \\
&f_h \int_0^x R_h'(z)dz \leq x + \Delta, && \forall x \in X \\
&R_h(0) = 0
\end{aligned}$$

Using the same arguments as that in the proof for Proposition 1, we can deduce that the solution to the above problem is a debt contract given by:

$$R_h(x) = \begin{cases} \frac{x+\Delta}{f_h} & \text{if } x \leq x^* \\ \frac{x^*+\Delta}{f_h} & \text{otherwise} \end{cases}$$

where x^* is the solution to:

$$\int_0^{x^*} g_h(1-H(z))\,dz = c.$$

The above security is equivalent to a direct reimbursement of Δ coupled with the same debt contract as described in Proposition 1.

Case 2: Suppose the issuer raises $c+\Delta$ to fund a project whose return is governed by the distribution $H(x)$, and both investor types participate in the optimal mechanism. Analogous to Case 1, this can be reinterpreted as an alternative scenario where the designer intends to raise $c+\Delta$ to fund a project whose return is given by the distribution $H(x-\Delta)$. Technically, given that Lemma 2 remains valid, we obtain that $t_h = 1$ and that $t_l = \frac{c-f_h+\Delta}{f_l}$. The formulation of the designer's problem, excluding (IC-l) and

(IR-h), becomes then:

$$
\begin{aligned}
&\min_{(R_\theta,t_\theta)_{\theta\in\Theta}} \left\{ \int_X [f_l R_l'(x) + f_h R_h'(x)](1-H(x))dx \right\} \\
&\quad \text{s.t.} \int_X R_l'(x) g_l (1-H(x))dx = t_l = \frac{c - f_h + \Delta}{f_l} \\
&\quad\quad \int_X \left[R_h'(x) - R_l'(x)\right] g_h (1-H(x))dx = t_h - t_l = 1 - \frac{c - f_h + \Delta}{f_l} \\
&\quad\quad R_h'(x), R_l'(x) \geq 0,\ \forall x \in X \\
&\quad\quad f_l \int_0^x R_l'(z)dz + f_h \int_0^x R_h'(z)dz \leq x + \Delta,\ \forall x \in X \\
&\quad\quad f_l R_l(0) + f_h R_h(0) \leq \Delta
\end{aligned}
$$

By using the same arguments as that in the proof for Theorem 2, we obtain that the solution to the above problem is given by

$$
R_l^*(x) = \begin{cases} \frac{x+\Delta}{f_l} \text{ for } x \leq x_l^* \\ \frac{x_l^*+\Delta}{f_l}, \text{ otherwise} \end{cases}
$$

$$
R_h^*(x) = \begin{cases} 0, \text{ for } x \leq x_l^* \\ \frac{x-x_l^*}{f_h}, \text{ for } x_l^* \leq x \leq x_h^* \\ \frac{x_h^*-x_l^*}{f_h}, \text{ otherwise} \end{cases}
$$

In the above formulas, x_l^* and x_h^* are defined as in Theorem 2. Analogous to Case 1, the present situation can be interpreted as equivalent to directly reimbursing the low type agents with the additionally raised amount Δ, followed by issuing to both types the same securities as in Theorem 2.

By the above, we can conclude that it is without loss to solely concentrate on asset-backed securities.

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