Optimal Auctions: Non-Expected Utility and Constant Risk Aversion*

Alex Gershkov, Benny Moldovanu, Philipp Strack and Mengxi Zhang†

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Abstract

We study auction design for bidders equipped with non-expected utility preferences that exhibit constant risk aversion (CRA). The CRA class is large and includes loss-averse, disappointment-averse, mean-dispersion and Yaari's dual preferences as well as coherent and convex risk measures. The optimal mechanism offers "full-insurance" in the sense that each agent's utility is independent of other agents' reports. The seller excludes less types than under risk neutrality, and awards the object randomly to intermediate types. Subjecting intermediate types to a risky allocation while compensating them when losing allows the seller to collect larger payments from higher types. Relatively high types are anyway willing to pay more, and their allocation is efficient.

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1 Introduction

Auctions create risks for bidders because both the allocation of physical goods and the associated payments are ex-ante random. Sellers facing risk averse bidders can earn more profit (relative to the risk neutral benchmark) by exploiting their unwillingness to undertake risks. Maskin and Riley [1984] studied optimal auction design with risk averse bidders that maximize expected utility. In that case, the celebrated payoff and revenue equivalence results that hold for the risk-neutral case fail, and the optimal auction format crucially depends on bidders' risk preferences.

In most auctions the stakes are small or moderate compared to the total wealth of the involved agents. Plausible calibrations of expected utility theory generally lead to risk-neutral behavior over small stakes (see, e.g., Rabin [2000]). But, there is ample field and laboratory evidence that the expected utility theory does not perform well in explaining agents' risk attitude over small or modest stakes.

A large and important class of models within the framework of non-expected utility (non-EU) generates *first-order risk aversion* and can provide a more plausible account of modest-scale risk attitudes (see Kahneman and Tversky [1979], Yaari [1987], Quiggin [1982] and Gul [1991] among others). In addition, very large theoretical and applied literature in finance, bank regulation and insurance define and analyze *risk measures* that are directly derived from these non-EU decision-theoretic models (see, for example, the excellent textbook by Föllmer and Schied [2011]).

In this paper, we derive the revenue maximizing mechanism for a seller facing risk-averse bidders endowed with non-expected utility preferences exhibiting a constant attitude towards risk. The informational assumptions in our model are otherwise standard, and follow the independent, private values paradigm.

Constant risk aversion (CRA) means that adding the same constant to all outcomes of two lotteries, or multiplying all their outcomes by the same positive constant, will not change the preference relation between them. These properties make constant risk aversion rather appealing for auction settings where stakes are small or moderate.

While CRA reduces to risk neutrality within the expected utility framework (i.e., in that case our analysis reduces to the classical one due to Myerson [1981] and Riley and Samuelson [1983]), it yields a novel and rich framework for mechanism design once the expected utility (EU) hypothesis is dropped.

The building blocks of Safra and Segal's [1998] characterization of CRA preferences are Yaari's *dual utility* functionals (Yaari [1987]). These functionals are obtained by applying a distortion (or weight function) to the probabilities attached to various events, and they allow a separation between risk attitudes and wealth effects.[1] Any risk averse CRA utility can be obtained as a minimum over such functionals, each of them being characterized by its respective convex distortion of probabilities. Examples include Yaari's dual utility itself, Gul's [1991] and Loomes and Sugden's [1986] disappointment aversion theories with a linear utility over outcomes, Köszegi and Rabin's [2006] loss-averse utility with a linear utility over outcomes, mean-dispersion utility of the type used in the macro and finance literature (e.g., Rockafellar et al [2006] and Blavatskyy [2010]), and maxima or minima over any sets of CRA utility functionals (see Appendix A). Moreover, CRA utilities are mirror images of the large class of *coherent* and *convex risk measures* appearing in the financial literature mentioned above. For example, Yaari's dual utility functionals correspond to the so-called distortion (or spectral) risk measures, related to the *weighted average values at risk* (see Föllmer and Schied [2011] and Rüschendorf [2013]).

The main structural feature that distinguishes the present preferences from expected utility and that is responsible for the novel properties of optimal mechanisms in our framework is *first-order risk aversion* (see Segal and Spivak [1990]): even in the limit where the stakes become infinitesimally small, our risk-averse bidders are willing to pay a strictly positive risk premium in order to avoid an actuarial fair risk. In contrast, it is well known that any EU preference represented by a twice differentiable utility function exhibits *second-order risk aversion:* in the limit where the stakes become small, EU agents become risk neutral and the risk premium they require tends to zero. This phenomena can have far-reaching implications for behavior. For example, Epstein and Zin [1990] found that Yaari's dual utility can partially resolve the equity premium puzzle posed by Mehra and Prescott [1985]: faced with lotteries with small stakes, a dual risk-averse (EU risk-averse) agent requires a risk premium proportional to the standard deviation (variance) of the lottery. For small risks, the standard deviation is considerably larger than the variance and can thus generate a higher equity premium.

In their study of revenue maximization for risk averse, expected-utility bidders, Maskin and Riley [1984] noted that a risk neutral seller faces a trade-off between "the desirability of insuring buyers against risk, and the desirability of exploiting their risk-bearing in order to screen them". As smooth EU risk aversion is of second-order, the bidders' incentives to pay for insurance vanish as the degree of risk exposure goes to zero, while the screening concern always remains relevant. Thus, screening eventually becomes the dominant factor, and full-insurance cannot be optimal.

[1]This class of preferences is also obtained as a special case of Quggin's [1982] well-known model of *rank dependent utility*, where the utility function evaluating outcomes is linear.

In contrast, because the preferences studied in this paper exhibit first-order risk aversion, the bidders' willingness to pay for insurance remains positive even in the limit: our first main result (Proposition 2) shows that, under a very weak form of risk aversion, the search for an optimal procedure can be confined to the class of incentive compatible (IC) full-insurance mechanisms, where the utility of an agent is only a function of his type: it depends neither on the types of other agents, nor on the realization of other randomizations within the mechanism. In particular, this means that losing buyers must be compensated in order to make them indifferent to winning. Proposition 3 shows that any allocation function that induces for each bidder a monotonic expected probability of getting the object is implementable by a full insurance mechanism.[2]

It is important to note that incentive compatible, full-insurance mechanism still involves risks for agents that deviate from truth-telling. Our second main result Corollary 1 derives upper and lower bounds for the expected revenue in any full-insurance mechanisms. These bounds are functions of the agents' limit risk premia required for binary lotteries when deviations from truth-telling become infinitesimally small. In particular, this result implies that the same revenue can be obtained from bidders with different preferences that nevertheless agree on a small set of binary lotteries. For the classical risk-neutral case (which is included as a special case), these bounds reduce to an instance of payoff equivalence (see Myerson [1981]). The same technique of using the order of risk aversion shows that the revenue obtained by a full insurance mechanism never exceeds the revenue from a second-price auction among bidders with smooth EU risk aversion. Since Maskin and Riley [1984] have proven that, under very general conditions, second price auctions are not optimal with EU risk averse bidders, the above observation yields that full insurance mechanisms cannot be generally optimal with EU risk-aversion.

Armed with the above insights, we turn to revenue maximization. The idea is to construct an incentive compatible, full-insurance mechanism that achieves the upper revenue bound derived in Corollary 1. This bound is a non-linear function of the reduced-form allocation (i.e., via the risk premium), and the maximization exercise must be approached by convex analysis/optimal control methods.

Consider first the optimal mechanism for a single bidder: instead of a classical take-it-or-leave offer for a risk-neutral buyer, we find that the seller awards the object randomly to intermediate types. Subjecting these types to a risky lottery while compensating them when they do not get the object allows the seller to collect larger payments from higher types, which is ultimately profitable. High types receive the object with probability one,

[2]Note that the converse need not hold here!

as distorting their allocation is too costly. This is related to a monopolistic screening problem a la Mussa and Rosen [1978][3]: even if the cost of producing any quality is here zero, it is sometimes optimal for the monopolist to sell intermediate types a "damaged" good (that sometimes malfunctions) plus a full-insurance warranty.

The main complication of the n-bidders allocation problem relative to the single-bidder case is the feasibility constraint, binding across types, that restricts the reduced form allocation, i.e., the expected probability of obtaining an object for each bidder type (see Border [1991]).[4] Incorporating this constraint yields an optimal control problem with a *pure state constraint*. Under a regularity condition that generalizes the standard monotonicity of the virtual value function, we solve this problem (see Theorem 1) for any utility that agrees with a Yaari functional on binary lotteries, e.g., disappointment-averse or loss-averse preferences with a linear utility over outcomes, and mean-dispersion preferences as used in the macro-finance literature. The optimal allocation has features similar to that in the single-bidder case: in particular, even when the object is allocated, it is not always allocated efficiently, and payments are computed to yield full insurance.

Finally we discuss the general case with constant risk aversion and show that the induced optimal control problem has a solution and that the necessary conditions for optimality are also sufficient (Theorem 2). Moreover, the expected revenue increases when bidders become more risk-averse.

In Appendix A we list several well-known non-expected utility preferences that satisfy our assumptions and their relations to Yaari functionals on binary lotteries. All proofs are in Appendix B.

1.1 Related Literature

Most of the papers investigating auctions with EU risk-averse bidders (e.g. Matthews [1987], Baisa [2017]) do not aim to provide a characterization of the optimal mechanism. Revenue maximization with risk averse buyers under expected utility has been studied by Maskin and Riley [1984] and by Matthews [1983]. Matthews [1983] restricts attention to constant absolute risk aversion (CARA) expected utility preferences and finds that the optimal mechanism resembles a modified first-price auction where the seller sells partial insurance to bidders with high valuation, but charges an entry fee to bidders with low valuation. The optimal auction in the EU case with CARA shares with ours the property

[3] This problem was analyzed by Matthews and Moore [1987] for a risk-averse agent with EU preferences.

[4] The basic maximization problem is here concave. See Gershkov et al. [2019] for the analysis of a convex revenue maximization problem (with the same obstacle) via the Fan-Lorentz integral inequality.

that intermediate types can get a random allocation. Note that Matthews' derivation holds for one special functional form of utilities - an exponential. Although we impose, in addition, constant relative risk aversion (CRRA) our treatment of the non-EU case holds for a very large class of different utilities.

Baisa [2017] studies an auction model with non-quasilinear and risk averse preferences and finds that standard auctions that allocate the good to the highest bidder are no longer revenue maximizing: the designer may prefer to utilize lotteries[5]. He introduced a novel mechanism, the *probability demand mechanism*, and showed that it outperforms all standard auctions, but does not characterized the optimal mechanism in his framework.

Maskin and Riley [1984] allow for more general risk averse (EU) preferences and establish several important properties of an optimal auction. In particular, they show that full insurance need not be optimal in their expected-utility framework. We explain and contrast this result with ours.

It is interesting to point out that Riley and Samuelson [1983] have constructed full insurance mechanisms (which they called *Santa Claus auctions*) that maximize revenue in the EU risk-neutral case. Of course, in that case, many other "standard" mechanisms are also optimal. The optimality of offering insurance also appears in the context of auctions with ambiguity-averse bidders: Bose et al. [2006] consider the two-bidder, risk-neutral case and show that an insurance mechanism is optimal among all deterministic mechanisms. In their framework, bidders are indeed insured against variations in the other bidder's type, but not necessarily against the allocative risk generated by the mechanism itself or by ambiguity within the mechanism (the risk coming from random transfers is not relevant as they restrict attention to deterministic mechanisms). Their induced maximization problem is linear in probabilities, and thus the respective optimal auction is obtained by standard methods.

Analogously to the EU case, almost all papers studying auctions where bidders have non-expected utility typically compare the performance of specific selling formats, e.g., the early contributions of Neilson [1984], Karni and Safra [1989], Lo [1998], and, more recently, Che and Gale [2006]. This last paper shows that, for a large class of non-expected utility risk-averse preferences, a first price auction yields a higher revenue than a second-price auction. None of these authors discussed optimal mechanisms in their respective frameworks. Heidhues and Köszegi [2014] consider a profit-maximizing monopolist selling to a single representative consumer who is loss averse. They show how randomized prices

[5]See also Kazumura, T., Mishra, D., & Serizawa [2020] who study a single-agent mechanism design problem with non-quasilinear preferences and show that revenue equivalence fails.

can increase revenue by affecting the buyer's reference point for evaluation her purchase.

Volij [2002] studied standard auctions where bidders' preferences follow Yaari's dual utility (recall that these preferences are the building blocks for the class of CRA preferences analyzed here), and claimed a payoff-equivalence result within this class. Unfortunately, his result is not correct[6]. We explain the problem in more detail after the derivation of the revenue bounds in Proposition 3.

A well-known example of auctions where (some) losers are compensated are the so-called *premium* auctions where the seller rewards one, or more, high losing bidders[7]. Although in practice not all losers who bid above a threshold are compensated (as would be required in our full insurance mechanisms), the implied compensation in our mechanisms for low and intermediate types is relatively small because their chance of winning is also small. Thus, in our optimal mechanism only high-type losing bidders get substantial compensation, which is broadly consistent with the practice of premium auctions. Milgrom [2004] and Goeree and Offerman [2004] suggest that a premium auction format is used to encourage weak bidders to compete against strong bidders. Hu, Offerman and Zou [2011] studied a two-stage English premium auction model with symmetric, interdependent values, and showed that the use of premium is only profitable to the seller when bidders are risk-loving. Hu, Offerman and Zou [2017] showed that, if both the seller and the bidders are risk-averse, premium auctions allow risk sharing that may benefit all participants. All identified reasons where premium auctions may be beneficial are thus quite different from our insights.

We note that full insurance contracts are also consistent with observations from real-life insurance markets where even moderate risks are often fully insured: Cohen and Einav [2007] (house insurance) and Sydnor [2010] (car insurance), among others, empirically show that assuming EU yields implausibly large measures of risk parameters for a range of moderate risks. Most customers in their studies purchase low deductibles - de facto warranties - despite costs that are significantly above the expected value.

Finally, the observation that often agents' risk taking behavior cannot be rationalized by expected utility theory is also supported by evidence from sport bets studies (see Snowberg and Wolfers [2010]) and property insurance markets (Barseghyan et al [2011]).[8]

[6] We are extremely grateful to a referee who made us aware of this issue.

[7] Premium auctions have been around since the Middle Ages, and are still used today to sell houses, land, large equipment (e.g., boats, planes, machines) and inventories of insolvent businesses (see Goeree and Offerman [2004]).

[8] Looking at households that purchase property insurance Barseghyan et al [2011] reject the hypothesis that subjects have stable expected utility preferences for more than 3/4 of the households.This finding is confirmed for insurance coverage and 401(k) investment decisions in Einav et al [2012]. Barseghyan

Several laboratory experiments illustrated similar findings (see Bruhin, Fehr-Duda and Epper [2010] and Goeree, Halt and Palfrey [2002]).

2 The Auction Model

A risk-neutral seller has an indivisible object, and there are $n \geq 1$ potential buyers. The monetary valuation (or type) of bidder i for the object, $\theta_i \in [0,1]$, is drawn according to a distribution F_i with density $f_i > 0$, independently of other bidders' valuations.

In order to formally model both random allocations and random transfers that may depend on the realized allocation of the object, it will be useful to explicitly specify the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which these random variables are defined. We assume that all randomness in the mechanism is derived from a single random number $r \in [0,1]$ that is drawn in addition to the draws of individual types.[9] We denote by $\omega = (\boldsymbol{\theta}, r) \in \Omega = [0,1]^{n+1}$ a realization of types and of the random number. We denote by $\mathbb{P}[\cdot]$ the probability measure on Ω defined by drawing θ_i independently according to F_i, and r independently from the uniform distribution on $[0,1]$. We denote by $\mathbb{E}[\cdot]$ the associated expectation operator.

2.1 Preferences

Auction mechanisms (see below for the formal definition) induce lotteries that specify for each bidder a probability of getting the object - and hence of receiving the associated monetary valuation specified by the bidder's type - and, in addition, a possibly random monetary payment that the bidder must make. Thus, to describe bidders' behavior we need to first specify their preferences over lotteries with monetary payoffs.

Let X be the set of bounded random variables defined on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution of a random variable x is denoted by F_x. With a slight abuse of notation, we let b represent a constant random variable with value $b \in \mathbb{R}$.

We assume that each bidder i's preferences $\succeq_i$ can be represented by an utility functional $\mathcal{U}_i$ defined on X that satisfies the following basic properties.

Continuity: $\mathcal{U}_i$ is continuous with respect to the weak topology.

Law Invariance: For any $x, y \in X$, $F_x = F_y \Rightarrow \mathcal{U}_i(x) = \mathcal{U}_i(y)$.

et al [2016] also find that stable Yaari and rank-dependent utility preferences cannot be rejected for the majority of households in a data set of car and home-insurance choices.

[9] This is without loss of generality by the general results of Halmos and von Neumann [1942].

Monotonicity: For any $x,\, y \in X$, $x \geq y$ a.s. implies $\mathcal{U}_i(x) \geq \mathcal{U}_i(y)$. [10]

The above are standard properties, satisfied by practically all preference relations used in applications. In particular, they imply the existence of a unique certainty equivalent $CE_i(x) \in \mathbb{R}$ such that $\mathcal{U}_i(x) = \mathcal{U}_i(CE_i(x))$ for each $x \in X$.

In addition, for our specific results we assume the following substantial properties[11]:

A1 Cash Invariance: For any $x \in X$ and any $b \in \mathbb{R}$ such that $x + b \in X$ it holds that

$$\mathcal{U}_i(x + b) = \mathcal{U}_i(x) + b.$$

A2 Positive Homogeneity: For any $x \in X$ and any $\alpha \in \mathbb{R}_+$ such that $\alpha x \in X$ it holds that $\mathcal{U}_i(\alpha\, x) = \alpha \mathcal{U}_i(x)$.

A3 Diversification: For any lotteries $x, y \in X$ such that $\mathcal{U}_i(x) = \mathcal{U}_i(y)$ and for any $\alpha \in [0, 1]$ it holds that $\mathcal{U}_i\left(\alpha x + (1 - \alpha) y\right) \geq \mathcal{U}_i(x)$.

Assumptions A1 and A2 seems particularly relevant in auction settings where the value of the auction object is small relative to the bidders' wealth. Note that A1 and A2 imply together that $\mathcal{U}_i(b) = b$ for any $b \in \mathbb{R}$. Assumption A3 says that holding a portfolio of equally preferred lotteries allows some risk hedging, and hence yields some potential benefit over an un-diversified holding. By Dekel [1989] (see his Proposition 2) diversification implies *risk aversion* in the standard sense where $\mathcal{U}(x) \geq \mathcal{U}(y)$ if y is a *mean-preserving spread* of x[12].

Since for any random variable the constant variable equal to the mean is a mean-preserving contraction, risk aversion together with monotonicity imply that

$$\mathcal{U}_i(x) = \mathcal{U}_i(CE_i(x)) \leq \mathcal{U}_i(\mathbb{E}[x]) \Leftrightarrow CE_i(x) \leq \mathbb{E}[x].$$

Some of our results only require the above weak version of risk aversion, rather than diversification. Hence, for later use, we also introduce:

A3' Weak Risk Aversion: For any lottery $x \in X$

$$CE_i(x) \leq \mathbb{E}[x].$$

[10]In other words, U is consistent with first-order stochastic dominance.

[11]Not all our results require the full set of axioms. For each result we shall mention the set of axioms that is needed, respectively.

[12]Note that for EU maximizers diversification is equivalent to risk aversion.

Note that weak risk aversion is equivalent to consistency with second order stochastic dominance for Standard EU preferences, but this is not the case with non-expected utility preferences.

Remark 1: Analogs of the above axioms, conversely formulated for a functional $\rho(x) = -\mathcal{U}(x)$, play a major role in a very large finance literature on *coherent and convex risk measures* (see the excellent textbooks by Föllmer and Schied [2011] and Rüschendorf [2013]).

Remark 2: Assuming expected utility, the cash invariance axiom A1 focuses attention on the standard class of CARA utility functionals and Assumption A2 on the standard class of CRRA functionals. But, the only member of expected utility class that satisfies both CARA and CRRA is the one with linear utility - and hence with risk-neutrality. In other words, expected utility exhibiting strict risk aversion is not consistent with the above axioms. The situation completely changes in the framework of non-expected utility, where a large and very interesting class of utility functionals satisfies our axioms.

Definition 1 *Let $\phi : [0,1] \to [0,1]$ be increasing and onto. For each ϕ, the functional $F_x \to \int s \cdot d\phi(1 - F_x(s))$ is called a* Yaari functional*, and the utility given by $U_i(x) = \int s \cdot d\phi(1 - F_x(s))$ is called Yaari's* dual utility.[13]

Observe that risk aversion in Yaari's dual utility model corresponds to the convexity of ϕ.

Proposition 1 *(Safra and Segal [1998]) The following two conditions are equivalent:*

1. *$\mathcal{U}_i$ satisfies axioms A1-A3.*
2. *There exists a unique, compact set Φ_i of increasing, convex and onto functions ϕ over $[0,1]$ such that*
$$\mathcal{U}_i(x) = \min_{\phi \in \Phi_i} \int s \cdot d\phi(1 - F_x(s)).$$

Remark: Although our model does not contain any ambiguity, the above functional form resembles those appearing in the theory of maxmin utility under ambiguity (see the large literature following Gilboa and Schmeidler [1989]). The maximin utility model studies agents who behave as if they evaluate subjective uncertainty using the worst realization

[13]In the finance literature, this class is known under the name *distortion* or *spectral* risk measures and is related to the well-known *average value at risk* (see Rüschendorf [2013]).

of multiple priors, while our agents behave as if they evaluate objective risks by a worst-case scenario from a set of belief distortions. If, for example, the set of priors in the ambiguity model forms the core of a *convex capacity*, the resulting maxmin utility is also a Yaari functional and it can be treated by the methods of this paper. This is the case for the ε-contamination ambiguity model, studied in a two-person auction context by Bose [2006] ,which corresponds to the special case where, for $i = 1, 2$, $\varepsilon < 1$, $\phi_i(p) = \varepsilon p$ for $p < 1$ and $\phi_i(1) = 1$.

Within the framework of our other axioms, we could have assumed any one of the axioms below instead of diversification A3 in order to characterize the set of CRA preferences.

Lemma 1 *Assume that* $\mathcal{U}_i$ *satisfies Axioms A1-A3. Then* $\mathcal{U}_i$ *also satisfies*

A4 ***Super-additivity:*** *For any lotteries* x, $y \in X$ *it holds that* $\mathcal{U}_i(x+y) \geq \mathcal{U}_i(x)+\mathcal{U}_i(y)$.

A5 ***Concavity:*** *For any lotteries* $x, y \in X$ *and any* $\alpha \in [0,1]$ *it holds that*

$$\mathcal{U}_i\left(\alpha x + (1-\alpha)y\right) \geq \alpha\mathcal{U}_i(x) + (1-\alpha)\mathcal{U}_i(y).$$

2.2 Mechanisms

We restrict attention to direct mechanisms where each agent i only reports her type θ_i. This is without loss of generality even for agents with non-expected utility preferences as long as the designer is either restricted to static mechanisms, or as long as each agent is sophisticated and can commit to a strategy in the mechanism.[14] We make this assumption to rule out dynamic mechanisms that exploit the agents' time-inconsistency.[15]

A *direct mechanism* $(\boldsymbol{q}, \boldsymbol{t})$ specifies for each agent i an *allocation rule* $q_i : [0,1]^n \to [0,1]$ and a *transfer* $t_i : [0,1]^n \times [0,1] \to [-m, m]$.[16] We require both q_i and t_i to be measurable so that both allocation and transfer are well-defined random variables. To complete the description of the physical allocation, we define n non-overlapping sub-intervals of the

[14] Under either assumption, each type of an agent can commit to follow the strategy of another type. This means that incentive compatibility of the original mechanism implies incentive compatibility of the direct mechanism implementing the same allocation and transfers. This is, for example, discussed in Bose & Daripa [2009].

[15] See Machina (1989) for an excellent discussion of this issue.

[16] We need to impose an upper bound on the transfer to ensure that a bidder's utility is bounded from below so that her preferences are well defined. But, this upper bound can be arbitrarily large, and thus imposes no economically meaningful restriction.

unit interval, one for each agent i, by

$$W_i(\boldsymbol{\theta}) = \left[\sum_{j=1}^{i-1} q_j(\boldsymbol{\theta}), \sum_{j=1}^{i} q_j(\boldsymbol{\theta})\right) . \tag{1}$$

Agent i receives an object if and only if $r \in W_i(\boldsymbol{\theta})$, and pays the transfer $t_i(\boldsymbol{\theta}, r)$. Note that, conditional on the vector of types $\boldsymbol{\theta}$, the probability with which agent i receives the good, is

$$\mathbb{P}[r \in W_i(\boldsymbol{\theta}) \mid \boldsymbol{\theta}] = q_i(\boldsymbol{\theta}) .$$

Furthermore, for any realization of $(\boldsymbol{\theta}, r)$, at most one agent receives the object. Note also that, since it depends on the random number r, the transfer t_i of agent i may be random (even conditional on the agents' types $\boldsymbol{\theta}$ and on the allocation of the good!).

Fix now a mechanism $(\boldsymbol{q}, \boldsymbol{t})$. Let $u_i(\theta_i, \theta_i', \boldsymbol{\theta}_{-i}, r_i) : [0,1]^{n+2} \to [-m, 1+m]$ denote the *ex-post payoff* of agent i with type θ_i who reports that he has type $\theta_i^{'}$ while all other agents report types $\boldsymbol{\theta}_{-i}$. We slightly abuse notation by using $u_i(\boldsymbol{\theta}, r) = u_i(\theta_i, \boldsymbol{\theta}_{-i}, r_i)$ instead of $u_i(\theta_i, \theta_i, \boldsymbol{\theta}_{-i}, r_i)$ for the case where agent i is truthful.

Let $V_i(\theta_i, \theta_i')$ denote agent i's certainty equivalent assuming that all agents other than i report truthfully, and that agent i has type θ_i, but reports type θ_i'. We again slightly abuse notation by using $V_i(\theta_i)$ instead of $V_i(\theta_i, \theta_i)$.

A mechanism $(\boldsymbol{q}, \boldsymbol{t})$ is *incentive compatible* if, for each agent i and for each pair of types θ_i and $\theta_i' \neq \theta_i$, it holds that:

$$V_i(\theta_i) = V_i(\theta_i, \theta_i) \geq V_i(\theta_i, \theta_i').$$

Whenever we want to keep track of a mechanism that varies, we shall also use the notation $V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t})$ instead of $V_i(\theta_i)$.

3 Full-Insurance Mechanisms

Our first main result shows that, in order to search for the seller-optimal mechanism, we can restrict attention to full-insurance mechanisms.

Definition 2 *A full-insurance mechanism is one where the ex-post payoff of any bidder i with type θ_i who truthfully reports his own type is a constant. That is, $(\boldsymbol{q}, \boldsymbol{t})$ is a full insurance mechanism if and only if, for all i and all θ_i, $u_i(\theta_i, \boldsymbol{\theta}_{-i}, r)$ does not depend on $(\boldsymbol{\theta}_{-i}, r)$.*

The superiority of full insurance mechanisms is very general: the proof of the Proposition below uses only cash invariance (A1), superadditivity (A4) and weak risk aversion (A3').

Proposition 2 *For any incentive compatible mechanism* $(\boldsymbol{q}, \boldsymbol{t})$, *there exists an incentive compatible, full-insurance mechanism that implements* $\boldsymbol{q}$ *and the same (non-expected) bidder utilities, and that is at least as profitable for the seller.*

The incentive compatibility of the constructed full insurance mechanism follows from cash invariance and superadditivity: it is always more costly for agents to deviate from a constant payoff than from a lottery with the same certainty equivalence. Since the seller is risk neutral and the bidders are (weak) risk-averse, full-insurance mechanism maximizes the total social surplus and thus leaves more revenue to the seller.

3.1 Implementable Mechanisms and Revenue Bounds

For any allocation rule $\boldsymbol{q}$, define

$$Q_i(\theta_i) = \mathbb{E}[q_i(\theta_i, \boldsymbol{\theta}_{-i}) \mid \theta_i]$$

to be bidder's i induced interim probability of obtaining an object, given that he is of type θ_i and that he reports it truthfully. Observe that, by the law of iterated expectations, $Q_i(\theta_i)$ equals the interim probability $\mathbb{P}\left[r \in W_i(\theta_i, \theta_{-i}) \mid \theta_i\right]$ assigned by agent i to the event where he receives an object after observing his type θ_i. These expected probabilities are called *reduced form* allocations by Border [1991].

A reduced form allocation $\boldsymbol{Q} = (Q_1, Q_2, ...Q_n)$ is *feasible* if there exists an allocation function $\boldsymbol{q}$ that induces it. A feasible $\boldsymbol{Q}$ is *implementable* (or *incentive compatible*) if there exists an incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$ such that $\boldsymbol{q}$ induces $\boldsymbol{Q}$.

Definition 3 *Assume that agent ihas a CRA Utility function* $\mathcal{U}_i$ *and let* $x(1,p)$ *denote a binary lottery that yields* 1 *with probability* p *and yields* 0 *otherwise. We define*

$$g_i(p) = \mathcal{U}_i(x(1,p)) = \min_{\phi \in \Phi_i} \phi(p).$$

In words, the function g_i is the certainty equivalent of a binary lottery that yields 1 with probability p and zero otherwise. It describes the probability distortion used by the agent to assess simple binary lotteries. For example, we obviously have $g_i(p) = \phi(p)$

for any Yaari risk-averse utility functional represented by the probability distortion ϕ, and hence g_i is then convex. It is important to note that g_i is convex for many other well-known utility functions (see Appendix A) that differ from Yaari's, but nevertheless coincide with it on binary lotteries.

We show below that bidders' utilities from a mechanism and hence also the revenue obtained by the seller from each bidder can be characterized by the function $g_i(p)$. In particular, only very minimal information about utilities is needed to compute the optimal auction.

The following result is very general and holds for any preferences satisfying cash invariance (A1), positive homogeneity (A2) and weak risk aversion (A3').

Proposition 3 *Let* $(\boldsymbol{q}, \boldsymbol{T})$ *be a full-insurance mechanism and* $\boldsymbol{Q}$ *be the reduced form allocation rule induced by* $\boldsymbol{q}$. *The mechanism* $(\boldsymbol{q}, \boldsymbol{T})$ *is incentive compatible only if, for all* i, $V_i(\theta_i)$ *is Lipschitz-continuous and satisfies*

$$g_i\left(Q_i(\theta_i)\right) \leq V_i'(\theta) \leq 1 - g_i\left(1 - Q_i(\theta_i)\right).$$

If each component of $\boldsymbol{Q}$ *is non-decreasing, then the above condition is also sufficient.*

In a full insurance mechanism, a bidder who deviates from truth-telling is subjected to risk. If a type θ_i bidder pretends to be of type θ_i', he obtains a constant payoff of $V_i(\theta_i')$ and, in addition, a lottery which gives him a payoff of $\theta_i - \theta_i'$ with probability $Q_i(\theta_i')$. The rank of the payoffs varies however: under-reporting generates a lottery with a possible positive payoff, while over-reporting generates one with a possible negative one. As our agents overweigh negative scenarios, they are more averse to negative risks. This asymmetry creates a range of payoffs that are incentive compatible: if the payoff from a full insurance mechanism increases faster than the upper bound, then agent has an incentive to pretend to be of a higher type; if the payoff increases at a slower speed than the lower bound, then the agent has an incentive to pretend to be of a lower type. Within the above range, the agent has no incentive to either over- or under-report.

Corollary 1 1. *Let* $(\boldsymbol{q}, \boldsymbol{T})$ *be a full-insurance mechanism that implements* $\boldsymbol{Q}$. *Then, the seller's expected revenue* $R(\boldsymbol{Q})$ *satisfies*

$$\begin{aligned}&\sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - \frac{1 - F_i\left(\theta_i\right)}{f_i\left(\theta_i\right)}\left[1 - g_i(1 - Q_i(\theta_i))\right]\right] f_i\left(\theta_i\right) d\theta_i - \sum_{i=1}^{n} V_i(0) \\ \leq & R(\boldsymbol{Q}) \leq \sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - \frac{1 - F_i\left(\theta_i\right)}{f_i\left(\theta_i\right)} g_i\left(Q(\theta)\right)\right] f_i\left(\theta_i\right) d\theta_i - \sum_{i=1}^{n} V_i(0).\end{aligned}$$

2. *For **any** incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$ that implements $\boldsymbol{Q}$, the seller's expected revenue $R(\boldsymbol{Q})$ satisfies*

$$R(\boldsymbol{Q}) \leq \sum_{i=1}^{n} \int_{0}^{1} \left[\theta_i Q_i(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} g_i(Q(\theta)) \right] f_i(\theta_i) \, d\theta_i - \sum_{i=1}^{n} V_i(0) \, .$$

The second part of the corollary follows from the first part together with Proposition 2.

Note that the classical risk-neutral model studied by Myerson [1981] and Riley and Samuelson [1981] is also included in the above corollary as a special case: the lower and upper bounds coincide then because $g(q) = q$, and we obtain an instance of the revenue equivalence result.

Remark: Consider the binary lottery $l = l(1, Q_i(\theta_i)) = x(1, Q_i(\theta_i)) - Q_i(\theta_i)$ that yields $1 - Q_i(\theta_i)$ with probability $Q_i(\theta_i)$ and $-Q_i(\theta_i)$ otherwise. By construction, it has zero mean. For any constant b (i.e., for any level of wealth), let $\pi(b, \varepsilon l)$ be the *risk premium* assigned to the combined lottery $b + \varepsilon l$. By the arguments used in the proof of Proposition 3, we obtain that, for any $Q_i(\theta_i) \neq 0, 1$, it holds that

$$\lim_{\varepsilon_i \searrow 0} \frac{\pi(b, \varepsilon l)}{\varepsilon} = Q_i(\theta_i) - g_i(Q_i(\theta_i)) > 0$$

if the agent is strictly risk averse. Thus, our preferences display *first degree risk aversion* (see Segal and Spivak [1990], and Quiggin and Chambers [1998]). In contrast, **any** expected utility preference with a smooth local utility displays *second degree risk aversion*: for any such utility preference, we have

$$\lim_{\varepsilon_i \searrow 0} \frac{\pi(b, \varepsilon l)}{\varepsilon} = 0 \text{ and } \lim_{\varepsilon_i \searrow 0} \frac{\pi(b, \varepsilon l)}{\varepsilon^2} > 0.$$

By arguments that are analogous to those used in the proof for Proposition 3, this observation yields:

Proposition 4 *Suppose that agent i has a smooth risk-averse expected utility preference represented by $\tilde{\mathcal{U}}_i$, and let $(\boldsymbol{q}, \boldsymbol{T})$ be a full-insurance mechanism that implements $\boldsymbol{Q}$. Then, the seller's expected revenue $\tilde{R}(\boldsymbol{Q})$ satisfies*

$$\tilde{R}(\boldsymbol{Q}) \leq \sum_{i=1}^{n} \int_{0}^{1} \left[\theta_i Q_i(\theta_i) - Q_i(\theta_i) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] f_i(\theta_i) \, d\theta_i - \sum_{i=1,2\ldots n} V_i(0) \, .$$

Remark: With symmetric agents and with a monotone hazard rate, the above upper bound is attained by an optimal second-price auction (with reserve price). That is, for risk-averse bidders with smooth expected utility, the revenue obtained by using a full insurance mechanism never exceeds the revenue from the optimal second-price auction. Under very general conditions, Maskin and Riley [1984] have proven that second-price auctions are not optimal with risk-averse bidders that have EU preferences. Together, the above observations imply that, under those general conditions, full insurance mechanisms cannot be optimal with EU risk-aversion.

Maskin and Riley [1984] did not discuss full insurance mechanisms as defined here. Instead, they studied *perfect insurance* mechanisms where bidders get the same *marginal utility* if they win or lose. Perfect and full insurance coincide for one special case, called Case 1 in their paper.[17] For that case, they show that a perfect (or full) insurance auction is revenue equivalent a second-price auction. This is consistent with our own finding above. In addition, we showed that, for all other cases, a full insurance auction is weakly less profitable than a second-price one.

4 Revenue Maximization

The main idea behind revenue maximization is to characterize an implementable allocation rule that achieves the upper bound of the seller's expected revenue (see Corollary 1) and then to display the transfer rule that can actually implements this bound. As our objective is to maximize the seller's revenue, it is optimal to always leave zero rent to the lowest type. Therefore, in the following analysis we only consider mechanisms where $V_i(0) = 0$ for all i.

4.1 The 1-Bidder Case

We drop here the subscript i and all respective variables refer here to a unique buyer. The allocation that achieves the upper bound of the seller's revenue solves

$$\begin{array}{lll} (P) & \max_{Q} \int_0^1 \left[\theta Q(\theta) - g(Q(\theta)) \frac{1-F(\theta)}{f(\theta)} \right] f(\theta) d\theta \\ s.t. & Q \in [0,1] \text{ and } Q \text{ is implementable.} \end{array}$$

[17] This is when there exists a concave and increasing function U such that: if an agent of type θ wins the object and pays t, his utility equals $U(\theta - t)$; if this agent does not win the object and pays t, his utility equals $U(-t)$.

Ignoring first the implementability constraint, let $Q^*(\theta)$ be the allocation rule that pointwise (e.g., for each θ) maximizes the principal's objective

$$P(\theta): \qquad Q^*(\theta) \in \arg\max_{p\in[0,1]} \left(\theta p - g(p)\frac{1-F(\theta)}{f(\theta)}\right).$$

Instead of the original problem $P(\theta)$ consider the *relaxed* problem

$$PR(\theta): \qquad Q^*(\theta) \in \arg\max_{p\in[0,1]} \left(\theta p - \bar{g}(p)\frac{1-F(\theta)}{f(\theta)}\right).$$

where $\bar{g}$ is the *convex bi-conjugate* (or convex envelope) of g, e.g., the highest convex function below g. Note that $\bar{g}(0) = 0$ and that $\bar{g}(1) = 1$. Several useful properties that hold for **any** increasing, convex function $\bar{g}$ on the interval $[0,1]$ such that $\bar{g}(0) = 0$ and $\bar{g}(1) = 1$ are: $\bar{g}$ is continuos on the interval $[0,1)$ and has well-defined one-sided derivatives $\bar{g}'_-$ and $\bar{g}'_+$ on $(0,1)$ that are non-negative and non-decreasing. Moreover, for any $p < \widehat{p}$ we know that $\bar{g}'_-(p) \leq \bar{g}'_+(p) \leq \bar{g}'_-(\widehat{p}) \leq \bar{g}'_+(\widehat{p})$ and that $\bar{g}'_-(p) = \bar{g}'_+(p) = \bar{g}'(p)$ almost everywhere, including at $p = 0$. In particular, any selection $\gamma(p) \in \partial\bar{g}(p) = [\bar{g}'_-(p), \bar{g}'_+(p)]$ is monotonic.[18]

The objective in $PR(\theta)$ is concave in p, and moreover, by the rules of conjugation for the sum of two functions one of them which is linear, it is also the lowest concave function above the objective function of $P(\theta)$. It is then well-known (see for example Hiriart-Urutty and Lemarechal [2013]) that p^* is a maximizer of $P(\theta)$ if and only it is a maximizer of $PR(\theta)$ such that

$$\theta p^* - \bar{g}(p^*)\frac{1-F(\theta)}{f(\theta)} = \theta p^* - g(p^*)\frac{1-F(\theta)}{f(\theta)}.$$

Whenever $\bar{g} \neq g$, the bi-conjugate $\bar{g}$ is affine, and thus the function $\theta p - \bar{g}(p)\frac{1-F(\theta)}{f(\theta)}$ is linear and cannot have a maximum in this set. Hence, the maximum in $PR(\theta)$ must be attained for p^* such that $\bar{g} = g$, and this p^* is also a maximizer of $P(\theta)$.

To conclude, if $Q^*(\theta)$ that solves $PR(\theta)$ is monotonic in θ, then it constitutes a solution to P. This will be the case, for example, if $\bar{g}'_-(1) < \infty$ and if $\theta - \bar{g}'_-(1)\frac{1-F(\theta)}{f(\theta)}$ is increasing.[19] To see that, consider a selection $\gamma(p)$ from the subdifferential $\partial\bar{g}(p) = [\bar{g}'_-(p), \bar{g}'_+(p)]$. As $\gamma(p) \leq \bar{g}'_-(1)$ for all $p \in [0,1]$, we obtain that $\theta p - \gamma(p)\frac{1-F(\theta)}{f(\theta)}$ is increasing in θ and supermodular in (θ, p), inducing the monotonicity of the maximizer Q^*.

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[18] See Hiriart-Urutty and Lemarechal [2013]

[19] Note how our regularity condition – monotonicity of $\theta - \bar{g}'_-(1)\frac{1-F(\theta)}{f(\theta)}$ – generalizes the usual "increasing virtual value" that obtains for $g(p) = p$. A sufficient condition for it to hold is that the condition holds for all convex functions in the set Φ_i that generates the agents' utility.

Let $(\gamma)^{-1}$ denote the pseudo-inverse of the monotonic selection from the subdifferential γ.[20] Defining

$$\begin{aligned} \theta_* &= \inf\left\{\theta \mid \bar{g}'(0) \leq \frac{\theta f(\theta)}{1-F(\theta)}\right\} \\ \theta^* &= \inf\left\{\theta \mid \bar{g}'_-(1) \leq \frac{\theta f(\theta)}{1-F(\theta)}\right\} \end{aligned}$$

we obtain that the optimal allocation is given by

$$Q^*(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_* \\ (\gamma)^{-1}\left[\frac{\theta f(\theta)}{1-F(\theta)}\right] & \text{if } \theta \in [\theta_*, \theta^*] \\ 1 & \text{if } \theta > \theta^*. \end{cases}$$

Although it seems a bit paradoxical to impose inefficient risks on a risk-averse buyer, randomization in this interval serves the purpose of weakening the incentive compatibility constraint: larger payments can be extracted from high types if low types are subjected to risk (see also Matthews [1983] for a similar feature of the optimal auction in the EU case with CARA utility). Thus, risk is imposed on a certain type - together with full insurance! - only to decrease the rent that must be paid to keep him from being mimicked by larger types that are not be fully insured if they deviate. In particular, it is unnecessary to impose much risk on a high type since it is unlikely that another buyer with an even higher type exists.

Technically, randomization should not be a surprise: contrasting classical results that considered with objectives that are linear or convex in probability where the maximum must be achieved on an extreme point (see Kleiner at al [2021]), the present problem is concave.

The revenue-maximizing transfers that implement this upper bound, T^{*w} (conditional

[20] Formally, $(\gamma)^{-1}(s) = \inf\{p \in [0,1] \mid \gamma(p) \leq s\}$.

on receiving an object), and T^{*l} (conditional on not receiving an object) are given by[21]

$$\begin{aligned} T^{*l}(\theta) &= \begin{cases} 0 & \text{if } \theta < \theta_* \\ -\int_{\theta_*}^{\theta} \bar{g}\left((\gamma)^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right) dt & \text{if } \theta \in [\theta_*, \theta^*] \\ -\int_{\theta_*}^{\theta^*} \bar{g}\left((\gamma)^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right) dt - g\left(1\right)\left(\theta - \theta^*\right) & \text{if } \theta > \theta^* \end{cases} \\ T^{*w}(\theta) &= \qquad \theta + T^l(\theta) \end{aligned}$$

the above defined mechanism $(Q^*, T^{*w}, T^{*l}(\theta))$ constitutes a solution to the seller's maximization problem.

Example 1 *Assume that the bidder's type is uniformly distributed on the interval* $[0,1]$ *and let* $g(p) = p^2$. *Then* $\frac{\theta f(\theta)}{1-F(\theta)} = \frac{\theta}{1-\theta}$ *and* $g'(p) = 2p$. *Therefore, the optimal allocation is given by:*

$$Q^*(\theta) = \begin{cases} \frac{1}{2}\frac{\theta}{1-\theta} & \text{if } \theta \in [0, 2/3] \\ 1 & \text{if } \theta > 2/3 \end{cases} .$$

The payments in case of losing and winning are given by

$$T^l(\theta) = \begin{cases} -\frac{1}{4}\left(\frac{(\theta-2)\theta}{\theta-1} + 2\log(1-\theta)\right) & \text{if } \theta \in [0, \frac{2}{3}] \\ -\theta + \frac{2}{3} - \frac{1}{4}\left(\frac{8}{3} - 2\log(3)\right) & \text{if } \theta > \frac{2}{3} \end{cases}$$

and

$$T^w(\theta) = \begin{cases} \theta - \frac{1}{4}\left(\frac{(\theta-2)\theta}{\theta-1} + 2\log(1-\theta)\right) & \text{if } \theta \in [0, \frac{2}{3}] \\ \frac{2}{3} - \frac{1}{4}\left(\frac{8}{3} - 2\log(3)\right) & \text{if } \theta > \frac{2}{3} \end{cases} .$$

The above finding can be contrasted to the optimal allocation and transfers for a risk-neutral bidder where $g(p) = p$. *This is given by:*

$$Q_r(\theta) = \begin{cases} 0 & \text{if } \theta \in [0, \frac{1}{2}] \\ 1 & \text{if } \theta > 1/2 \end{cases}$$

and

$$T_r^w(\theta) = \frac{1}{2} \quad \text{and} \quad T_r^l(\theta) = 0 \,.$$

In this case the revenue maximizing mechanism is deterministic, a take-it-or-leave-it offer

[21] We can arbitrarily define the losing payment $T^l(\theta)$ for $\theta > \theta^*$ (since the buyer gets then the object with probability 1), and the winning payment $T^w(\theta)$ for $\theta < \theta_*$ (since the buyer gets then the object with probability zero).

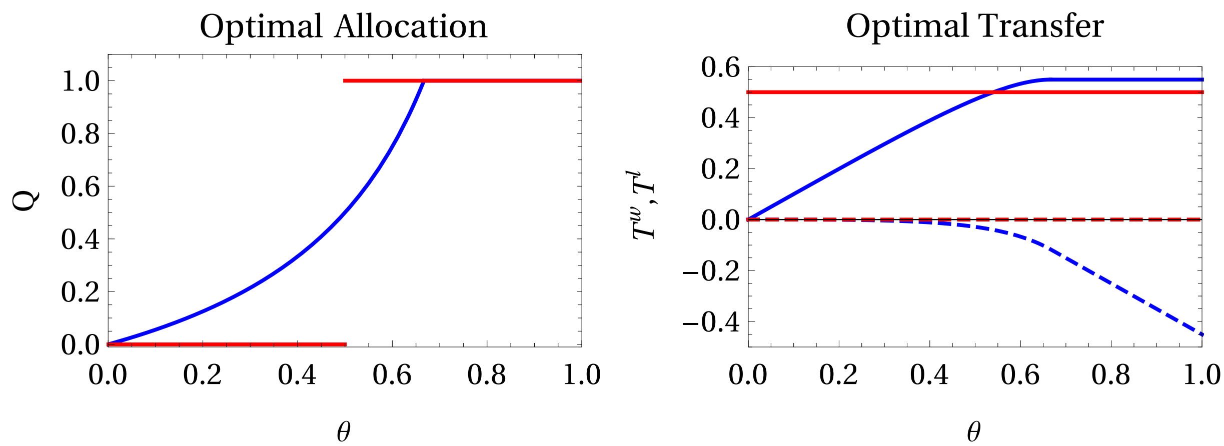

Figure 1: The optimal allocation and transfers in the risk averse case $g(p) = p^2$ (in blue) and the risk-neutral case $g(p) = p$ (in red) for θ uniformly distributed on $[0, 1]$. On the left is the probability of receiving an object as a function of the type. On the right is the transfer payed by the agent conditional on receiving an object (solid lines) and not receiving an object (dashed lines).

at a price $\widehat{\theta} = 0.5$ *(see Myerson [1981] or Riley & Zeckhauser [1983]). We illustrate the difference between the optimal mechanisms with and without risk aversion in Figure 1. Note that the take-it-or-leave-it scheme is incentive compatible even if the agent uses the present risk-averse preferences (as there is no uncertainty from the buyer's perspective). But, the seller increases her expected revenue by switching to the optimal mechanism we calculated above.*

Remark: Our model can be also interpreted as a monopoly screening model where the designer can choose different qualities and terms of trade for different types. The standard Mussa-Rosen model with a population of (expected utility) risk averse buyers has been analyzed by Matthews and Moore [1987] who interpreted quality as the probability of functioning. Matthews and Moore assume that the cost function is such that the monopolist never offers the highest quality (corresponding to functioning with probability one), and illustrate various properties of the optimal menu of offered qualities, prices, and warranties in case of malfunction. We showed above that, even if the cost of producing any quality is zero, in our model the monopolist sometimes provides a "damaged" good plus a full insurance warranty to intermediate types.

4.2 Revenue Maximization: The $n-$Bidder Case for Yaari's Dual Utility.

In this section we derive the revenue maximizing allocation for bidders with CRA preferences such that the certainty equivalent assigned to a binary lottery is convex in the probability of the good outcome. This condition is equivalent to a representation of preferences over binary lotteries being a Yaari functional with a convex probability weighting function. As noted before, many well-known utility theories are represented by functional forms that differ from Yaari's on the entire space of random variables but agree with Yaari's on the space of binary lotteries (see Appendix A for a partial list). Hence, the analysis in this Section applies to **any** CRA utility with the property that there exists an increasing and convex function $g : [0, 1] \to [0, 1]$ with $g(0) = 0$ and $g(1) = 1$ such that

$$\mathcal{U}(x(1, p)) = g(p)$$

where $x(1, p)$ is a binary lottery that yields 1 with probability p, and zero otherwise.

Example 2 *Consider mean-dispersion preferences with linear utility over outcomes (see Appendix A):*

$$\mathcal{U}(x) = \mathbb{E}(x) - \frac{1}{2} r \mathbb{E}\left[\mid x - \mathbb{E}(x) \mid\right]$$

where $r \in [0, 1]$. This formulation follows the logic of mean-variance preferences, but is modified to be consistent with FOSD. In case of a binary lottery this functional form coincides with a Yaari functional generated by

$$g(p) = p - rp(1 - p)$$

which is convex if $r > 0$. Our results below show that a seller facing bidders with mean-dispersion preferences obtains the same revenue as a seller facing bidders with the Yaari dual utility generated by g!

In this Section we assume that the setting is *symmetric* in the sense that all bidders share the same distribution of values $F_1 = F_2 = \ldots = F$ and the same preference over binary lotteries $g_1 = g_2 = \ldots = g$. In order to have a more transparent argument, we assume here that g is (everywhere) differentiable. The argument for the general case, where g is differentiable almost everywhere, is identical to the one given above in the 1-bidder case, and uses a monotonic selection from the subdifferential of g.

The seller's objective function

$$R = \sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - g(Q_i(\theta_i)) \frac{1 - F(\theta_i)}{f(\theta_i)} \right] f(\theta_i) \, d\theta_i$$

is concave in $(Q_i)_{i=1,2...n}$ because g is assumed here to be convex. Thus, without loss of generality, we can restrict our attention to symmetric mechanisms.[22]

The main complication relative to the 1-bidder case is the feasibility constraint (see Border [1991]) on the vector of reduced form allocations $\boldsymbol{Q} = (Q_1, Q_2, ..., Q_n)$. It is well-known that a necessary and sufficient condition for any symmetric, interim allocation $Q_i = Q$ to be feasible is that, for any subset of types $A \subset [0,1]$,

$$\int_A Q(t) f(t) dt \leq 1 - \left[\int_{\theta \notin A} f(t) dt \right]^n .$$

In words, the probability that any subset of types wins the object is never higher than the probability that a type in that set exist . We use below a simpler, relaxed version of the constraint that holds for any monotonic interim allocations Q, and later verify that the obtained solution is indeed monotonic and hence feasible. The seller's (relaxed) maximization problem over symmetric, implementable reduced-form allocation rules is:

$$\begin{aligned} &(R) \quad \max_{Q} n \int_0^1 \left[\theta p - g(p) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta, \\ &\text{s.t. (a) } Q \in [0,1]; \\ &\quad \text{(b) } Q \text{ is implementable} \\ &\quad \text{(c) } \int_\theta^1 Q(t) f(t) dt \leq \int_\theta^1 F^{n-1}(t) f(t) dt \quad \text{for any } \theta \in [0,1]. \end{aligned}$$

If a regularity condition holds, then the optimal mechanism is a full-insurance mechanism whose reduced form allocation consists of two parts: for lower types, the seller uses the same allocation rule - obtained by pointwise maximization - as in the single-buyer case. For higher types this becomes infeasible, and the seller allocates the object to the

[22] Consider, for example, the case of two bidders. Suppose there exists an optimal pair (Q_1^*, Q_2^*) such that $Q_1^* \neq Q_2^*$. By symmetry, (Q_2^*, Q_1^*) is also optimal. But then the symmetric allocation rule $(\frac{Q_2^*+Q_1^*}{2}, \frac{Q_2^*+Q_1^*}{2})$ is also feasible and it is at least as profitable for the seller (since R is concave in (Q_1, Q_2)). The generalization to more bidders is straightforward.

bidder with the highest type. To formally state the result, we define:

$$\begin{aligned}\theta_* &= \inf\left\{\theta \mid g'(0) \leq \frac{\theta f(\theta)}{1-F(\theta)}\right\} \\ \theta_n^* &= \inf\left\{\theta \mid g'(F^{n-1}(\theta)) \leq \frac{\theta f(\theta)}{1-F(\theta)}\right\}.\end{aligned}$$

Theorem 1 (Optimal Allocation) *Assume that the function*

$$\theta \mapsto \theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing almost everywhere in $[0,1]$*. Then, an optimal mechanism is a full-insurance mechanism that implements the following reduced form allocation rule:*

$$Q^*(\theta) = \begin{cases} 0 & \text{if } \theta < \theta_* \\ (g')^{-1}\left[\frac{\theta f(\theta)}{1-F(\theta)}\right] & \text{if } \theta \in [\theta_*, \theta_n^*] \\ F^{n-1}(\theta) & \text{if } \theta > \theta_n^* \end{cases}.$$

The transfers in the optimal mechanism (T^w, T^l) *conditional on winning and not winning an object are given by*[23]

$$\begin{aligned} T^l(\theta) &= \begin{cases} 0 & \text{if } \theta < \theta_* \\ -\int_{\theta_*}^{\theta} g\left((g')^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right)dt & \text{if } \theta \in [\theta_*, \theta_n^*] \\ -\int_{\theta_*}^{\theta_n^*} g\left((g')^{-1}\left[\frac{tf(t)}{1-F(t)}\right]\right)dt - \int_{\theta_n^*}^{\theta} g\left(F^{n-1}(t)\right)dt & \text{if } \theta > \theta_n^* \end{cases} \\ \text{and } T^w(\theta) &= \theta + T^l(\theta). \end{aligned}$$

Finally, the revenue in the optimal mechanisms increases in the number of bidders.

We note that the reduced form monotonic allocation $Q^* : [0,1] \to [0,1]$ satisfies by construction the feasibility condition (c) in Problem R above, and hence it can be generated as the marginal of an *allocation rule* $q_i = \mathbf{q}^* : [0,1]^n \to [0,1]$ (see Border [1991] or, more recently, Kleiner et al. [2021]). Together with the above constructed transfers (that only depend on own type and whether the bidder obtains the object or not) this yields an optimal direct mechanism, as desired.

Example 3 *Consider uniformly distributed types on the interval* $[0,1]$ *and let* $g_i(p) =$

[23] As in the one-bidder case we specify $T^w(\theta)$ for $\theta < \theta_*$. This transfer play no role in those cases.

$g(p) = cp^2 - cp + p$ with $0 < c < 1$. Then θ_* and θ_n^* solve

$$\frac{\theta_*}{1-\theta_*} = 1 - c; \ \frac{\theta_n^*}{1-\theta_n^*} = 2c\left(\theta_n^*\right)^{n-1} - c + 1,$$

Assuming $c = 1/2$ and $n = 3$, the regularity condition holds, and the optimal solution is given by Theorem 1 with $\theta_* = \frac{1}{3}$ and $\theta_3^* = 0.397$. It is important to note that, on binary lotteries, the utility associated with $g(p) = cp^2 - cp + p$ coincides with both the disappointment-averse preferences due to Loomes and Sugden [1986], and Jia et al. [2001], and to the modified mean-variance preferences with linear utility over outcomes analyzed by Blavatskyy [2010] (see Appendix A). Although these utilities do not generally coincide with Yaari's, the revenue maximization schemes for these classes of utility functions coincide with the one for the Yaari dual utility with the same g.

Example 4 *As mentioned earlier, the ε-contamination model studied by Bose et al [2006] corresponds to the case where $\varepsilon < 1$, $g(p) = \varepsilon p$ for $p < 1$ and $g(1) = 1$. It can be easily verified that, in this case, the two cutoff points θ_* and θ_n^* coincide, and hence the region of types where the optimal allocation is random vanishes. Thus, the optimal auction either assigns the object efficiently (high types) or does not assign at all (low types) - this is of course consistent with the finding of Bose et al [2006] in their model with ambiguity aversion.*

With a large number of bidders, the interval where the optimal allocation is random always vanishes:

Corollary 2 *Assume that*

$$\theta \mapsto \theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing in $[0,1]$. When $n \to \infty$, the interval where the optimal allocation is random, $[\theta_*, \theta_n^*]$, vanishes, and the limit optimal allocation rule assigns the object efficiently above a cutoff. The limit interval of excluded types $[0, \theta_*]$ is a subset of the interval of excluded types under risk neutrality, $[0, \theta_*^r]$.

The first statement follows because θ_n^* is non-increasing in n, and because $\lim_{n\to\infty} \theta_n^* = \theta_*$. The second statement follows because θ_*^r solves the equation $1 = \frac{\theta f(\theta)}{1-F(\theta)}$ (that is independent of n) and because $g'(0) \leq 1$. In particular, no type is excluded if $g'(0) = 0$. While the length of the intermediate interval, $[\theta_*, \theta_n^*]$, depends on the number of the bidders, the probability with which each intermediate type gets the object, $Q^*(\theta) = (g)^{-1}\left[\frac{\theta f(\theta)}{1-F(\theta)}\right]$, is independent of n.[24]

The needed randomization may be difficult to implement in practice since the seller needs commitment power. Imagine, for example, a realization where all bidders have intermediate types. Then, with positive probability, no bidder gets the object and all bidders get positive transfers - but the seller actually prefers to sell to a single bidder. It is of course difficult to ex-post verify that a randomization was performed with the pre-committed probabilities. Yet, recall that the optimal mechanism above was specified in terms of an interim randomization. As we know from Theorem 3 in Gutmann et. al. [1991], or from Chen et. al. [2019], there exists a feasible and deterministic allocation rule q^* with given marginals Q^*. This means that one can always achieve revenue maximization via a mechanism that does not involve any randomization.

In the risk-neutral case where $g(p) = p$, the above regularity condition reduces to the standard requirement that the virtual value $\theta - \frac{1-F(\theta)}{f(\theta)}$ is non-decreasing. Our final result in this Section shows that, under the increasing hazard rate condition, the regularity condition always holds as $n \to \infty$ if g'' is not too large.

Lemma 2 *Assume that $g'' < e$. Then, for any distribution F with an increasing hazard rate there exists $\bar{n}$ such that for all $n \geq \bar{n}$, the function*

$$\theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

is non-decreasing almost everywhere in $[0,1]$. *Hence, the monotonicity requirement of Theorem 1 holds.*

Example 5 *Köszegi and Rabin's [2006] loss-averse preferences*[25] *in the version with a linear utility over outcomes is given by*

$$\mathcal{U}(x) = \mathbb{E}(x) + \int\int \mu(x-y)\, dF(x)\, dF(y)$$

[24] The upper bound of the interval is the type whose optimal allocation coincides with the efficient allocation. Our regularity condition ensures the a crossing only happens once and that the optimal allocation crosses the efficient one from below. Thus, in this interval the optimal allocation always lies below the efficient allocation and any interim allocation rule that is strictly smaller than the efficient is feasible.

[25] We assume that the agents evaluate gains and losses based on the aggregate payoff, which is jointly determined by object allocation and monetary transfer.

where

$$\mu(z) = \begin{cases} z & \text{if } z \geq 0 \\ \lambda z & \text{if } z < 0 \end{cases} .$$

These preferences are risk averse and respect monotonicity (i.e. consistency with FOSD) if and only if $\lambda \in [1, 2]$. This functional form is then a special case of Yaari's dual utility where $g(p) = (2 - \lambda)p + (\lambda - 1)p^2$ (see Appendix A). Thus, we obtain that $g''(p) \leq e$ for ***any*** *λ in the relevant range.*

Similar computations can be made for the other examples in Appendix A.

4.3 Revenue Maximization for General CRA Utility Functions

In this section, we briefly discuss the more general case where $g(p) = \min_{\phi \in \Phi_i} \phi(p)$ is not necessarily convex. We follow Seierstadt and Sydsaeter [1987] (SS) Chapter 5, and formulate an optimal control problem while ignoring first the monotonicity constraint. For ease of reference, we use in this section the standard notation employed in the vast optimal control literature. We let the control be $u(t) = Q(t)$, and let the state be $x(t) = -\int_t^1 u(z)f(z)dz$. The symmetric revenue maximization problem becomes then[26]:

$$(P) \qquad \max_u \int_0^1 [tf(t)u(t) - g(u(t))(1 - F(t))]dt$$

subject to the constraints: 1. $u \in [0, 1] := U$; 2. $\dot{x} = u(t)f(t)$; 3. $\int_t^1 F^{n-1}(z)f(z)dz + x \geq 0$. The initial condition is $x(1) = 0$ and the terminal condition is $x(0) \in [-1, 0]$.

Because the control u does not appear in the feasibility constraint 3, this is a rather complex problem with a *pure state constraint.* The *Hamiltonian* for Problem (P) is:

$$H(u, p, t) = p_0[tf(t)u - g(u)(1 - F(t))] + p(t)uf(t)$$

and the *Langrangian* is:

$$\begin{aligned} \mathcal{L}(u, p, q, t) &= H(u, p, t) - q(t)[uf(t) - F^{n-1}(t)f(t)] \\ &= p_0[tf(t)u - g(u)(1 - F(t))] + p(t)uf(t) - q(t)[uf(t) - F^{n-1}(t)f(t)] \end{aligned}$$

where $p_0 \in \{0, 1\}$ and where p and q are multiplier functions[27].

[26]Note that, in general, the solution of a symmetric problem need not be symmetric because the problem is not concave anymore.

[27]For the last term in $\mathcal{L}$ observe that :

An important feature of our problem is that **neither** Hamiltonian **nor** Lagrangian depend on the state variable x. We assume below that $x(0) \in (-1, 0)$, i.e. the feasibility constraint 3 is not binding everywhere[28], which yields $p_0 = 1$. Very briefly, the main necessary conditions are:

1. Since $\frac{d}{dx}(\int_t^1 F^{n-1}(z)f(z)dz + x) = 1$, it must hold that

$$\frac{d(p^*(t) + q^*(t))}{dt} = -\frac{d\mathcal{L}}{dx} = 0\,.$$

   Hence, we obtain that

$$p^*(t) + q^*(t) = \text{constant}.$$

2. q^* is non-decreasing (and hence by the above condition p^* is non-increasing), and it is constant on any interval where the feasibility constraint is not binding:

$$\int_t^1 F^{n-1}(z)f(z)dz + x^*(t) > 0\,.$$

Theorem 2 *1. A solution (x^*, u^*) to (P) exists, and $u^* = Q^*$ is measurable.*

2. A solution also exists to the Problem (P) augmented by the constraint that u is monotonic.

3. Assume that u is monotonic and that a candidate (x, u) satisfies all necessary conditions (see Theorem 5.2 in SS [1987]). Then (x, u) is optimal.

For example, the proof of Theorem 1 for the special case where g is convex follows by setting $q^*(t) = \max\{0, t - g'(F^{n-1}(t)\frac{1-F(t)}{f(t)}\}$ and $p^*(t) = -q^*(t)$. The multiplier $q^*(t)$ is non-decreasing by the regularity assumptions made in that Theorem. Thus, the same assumption ensuring that q^* is non-decreasing also ensures that $u^* = Q^*$ is non-decreasing, and hence implementable. In this case sufficiency follows because the maximization problem is concave in the control u.

The solution to the problem when g is not convex needs to be constructed from the solution of a relaxed problem where g is replaced by its convex envelope. This is, in

---

$$uf(t) - F^{n-1}(t)f(t) = \frac{d}{dx}(\int_t^1 F^{n-1}(z)f(z)dz + x) \cdot \frac{dx}{dt} + \frac{d}{dt}(\int_t^1 F^{n-1}(z)f(z)dz + x)$$

where the last expression is the total derivative with respect to t of $\int_t^1 F^{n-1}(z)f(z)dz + x$.

[28] In that case the solution is simply gven by $u^* = F^{n-1}$.

principle, analogous to what we did in the 1-bidder case, but is much more complex because of the feasibility constraint[29].

We conclude by comparing the expected (optimal) revenues from agents with different risk-attitudes: the seller prefers to have bidders who are more risk averse. Recall that individual preferences are fully characterized by a set Φ of increasing and convex functions over $[0,1]$ (Proposition 1).

Definition 4 *An agent with preferences represented by the set Φ^1 is more risk averse (in a weak sense) than an agent with preferences represented by set Φ^2 if and only if for any $p \in [0,1]$ we have*

$$g^1(p) := \min_{\phi \in \Phi^1} \phi(p) \leq \min_{\phi \in \Phi^2} \phi(p) =: g^2(p).$$

The above definition is intuitive since it says that the agent with set Φ^1 requires a higher risk premium for binary lotteries than the agent with set Φ^2. For example, the property holds if $\Phi^2 \subset \Phi^1$ or if $\Phi^1 = \{g^1\}$, $\Phi^2 = \{g^2\}$ such that g^1 is a convex transformation of g^2. It implies that

$$\int s \cdot dg^1(1 - F_x(s)) \leq \int s \cdot dg^2(1 - F_x(s)).$$

In Yaari's dual utility case this definition suggests that the more risk averse agent is willing to pay a lower amount of money to purchase any given lottery than the less risk averse agent (See Definition 5 in Yaari [1986]).

Lemma 3 *Consider risk averse, CRA preferences represented by Φ^1 and Φ^2, respectively, and assume that $g^1(p) \leq g^2(p)$. Then the optimal expected revenue when facing agents with preferences represented by Φ^1 is higher than that obtained when facing agents with preferences represented by Φ^2.*

Example 6 *Consider Gul's disappointment-averse preferences with a linear utility over outcomes (see Appendix A). For binary lotteries, Gul's functional form is a special case of Yaari's preferences with $g(p) = \frac{p}{1+(1-p)\beta}$. The parameter β is strictly positive when the agent is strictly risk-averse. The above Lemma implies that the revenue of a seller who faces agents equipped with such utility functions is decreasing in β.*

Corollary 3 *The optimal revenue obtained when facing bidders with CRA risk-averse preferences represented by Φ is (weakly) smaller than the optimal revenue obtained when*

[29] For the solution to similar problems see Cellina and Colombo [1990] and Amar and Mariconda [1995].

facing bidders having a Yaari dual utility induced by the function $\bar{g}(p)$*, the convex bi-conjugate of* $g(p) := \min_{\phi\in\Phi^1} \phi(p)$.

Together with Theorem 1, the above Corollary offers an explicitly computable upper bound on the revenue that can be obtained from any risk averse bidders equipped with a given CRA utility function that satisfies the regularity condition imposed in that theorem. An analogous lower bound can be obtained by using for the preferences represented by Φ the optimal allocation for the Yaari dual utility induced by the convex function $\bar{g}(p)$.

5 Conclusion

We have derived the revenue maximizing mechanism in an auction framework where bidders have non-expected utility preferences that exhibit constant risk aversion of first order. For all the preferences in this large class, the feasible revenue is driven by the non-linear function governing the risk premium demanded by bidders on simple, binary lotteries. This yields a relatively complex variational problem, focused on the limited supply constraint. Our main results show how the optimal mechanisms bundles the allocation of the physical good with the sale of full insurance in order to increase revenue.

We expect our approach to be useful in many in other mechanism design frameworks where agents have various types of non-expected utility.

Appendix A: CRA Utility Functions

We briefly list and discuss here several well-known preferences classes that satisfy our axioms and hence are covered by our analysis. We show that the behavior induced by all these utility functions coincides with the behavior induced by a Yaari utility for the class of binary lotteries - this is the only set of lotteries that is relevant for our revenue derivations.

1. Gul's disappointment-averse preferences with linear utility over outocmes[30]:

$$\mathcal{U}(x) = \frac{\alpha}{1+(1-\alpha)\beta}\mathbb{E}[x|x \geq CE(x)] + \frac{(1-\alpha)(1+\beta)}{1+(1-\alpha)\beta}\mathbb{E}[x|x < CE(x)]$$

[30]Note that this standard representation is only implicit because it uses the simultaneously defined certainty equivalent. For a representation that avoids this see Cerreia-Vioglio, Dillenberger and Ortoleva [2020].

where $CE(x)$ is a certainty equivalent of lottery $x \in X$, α is the probability that the outcome of the lottery is above its certainty equivalent, and β is a parameter. Risk aversion corresponds to $\beta > 0$ (see Gul [1991], Theorem 3). Diversification for $\beta > 0$ follows by quasi-concavity (Proposition 3, Dekel [1989]) that in turn follows from the *betweenness* property of Gul's preferences. For binary lotteries, Gul's functional form above is a special case of Yaari's preferences: assuming a binary lottery with $x_1 > x_2$ we obtain

$$\mathcal{U}(x) = g(p)\, x_1 + (1 - g(p))\, x_2$$

where $g(p) = \frac{p}{1+(1-p)\beta}$ is convex in p, the probability of x_1.

2. Versions of the disappointment aversion theories due to Loomes and Sugden [1986], and Jia et al. [2001] with linear utility over outcomes:

$$\mathcal{U}(x) = \mathbb{E}(x) + (e\mathbb{E}[\max\{x - \mathbb{E}(x), 0\}] + d\mathbb{E}[\min\{x - \mathbb{E}(x), 0\}])$$

where $e > 0$, $d > 0$. We can rewrite the above as:

$$\mathcal{U}(x) = \mathbb{E}(x) + (e - d)\mathbb{E}[\max\{x - \mathbb{E}(x), 0\}]$$

Super-additivity follows from Theorem 3 in Rockafellar et al. [2006] if $d > e$. For a binary lottery with $x_1 > x_2$ we obtain

$$\mathcal{U}(x) = g(p)\, x_1 + (1 - g(p))\, x_2$$

where $g(p) = p(1+e-d) + (d-e)p^2$ that is convex in p, the probability of outcome x_1, if $d > e$. Hence, for binary lotteries, this is also a special case of Yaari's dual utility.

3. Köszegi and Rabin's [2006] loss-averse utility (see also the theory of disappointment without prior due to Delquié and Cillo [2006]). In the version with linear utility over outcomes (See Masatlioglu and Raymond [2016]) this is given by.

$$\mathcal{U}(x) = \mathbb{E}(x) + \int\int \mu(x - y)\, dF(x)\, dF(y)$$

where

$$\mu(z) = \begin{cases} z & \text{if } z \geq 0 \\ \lambda z & \text{if } z < 0 \end{cases}.$$

These preferences respect monotonicity (i.e. consistency with FOSD) if and only if $\lambda \in [0,2]$. This functional form is a special case of Yaari's dual utility with $g(p) = (2-\lambda)p + (\lambda-1)p^2$ (see Proposition 4 in Masatlioglu and Raymond[2016]). g is convex in p for $\lambda \geq 1$.[31]

4. Modified Mean-Variance preferences (see Blavatskyy [2010]) with linear utility over outcomes:

$$\mathcal{U}(x) = \mathbb{E}(x) - \frac{1}{2} r \mathbb{E}\left[\mid x - \mathbb{E}(x) \mid\right]$$

where $r \in [0,1]$. The modification relative to the standard mean-variance preferences is needed in order to ensure monotonicity (i.e., consistency with FOSD). Super-additivity follows from Theorem 3 in Rockafellar et al. [2006]. In case of a binary lottery with outcoms $x_1 > x_2$, this functional form is again a special case of Yaari's preferences:

$$\mathcal{U}(x) = g(p)x_1 + [1 - g(1-p)]x_2$$

where

$$g(p) = p - rp(1-p)$$

is convex in p for any $r > 0$.

Appendix B: Proofs

Proof of Lemma 1: Let $\Phi_i = \{\phi_i\}$ be a singleton, in which case $\mathcal{U}_i$ is a Yaari dual utility functional. For any comonotonic random variables $x, y \in X$ such that $x + y \in X$, an Yaari dual utility is additive (see Yaari [1987]):

$$\mathcal{U}_i(x+y) = \mathcal{U}_i(x) + \mathcal{U}_i(y)$$

Moreover, a Yaari functional U_i exhibits risk-aversion if and only if it is generated by a convex function ϕ_i. Meilijson and Nadas [1979] have shown that if the random vector $(x'_1, ..., x'_N)$ is comonotonic and has the same marginals as $(x_1, ..., x_N)$ then $\sum_{i=1}^N x'_i \geq_{cx} \sum_{i=1}^N x_i$, where cx denotes the convex stochastic order[32]. Together, these observations imply that if ϕ_i is convex (i.e., if the agent is risk averse) then for any two random variables

[31] For $\lambda = 1$ the model reduces to the standard EU risk neutral preferences.

[32] This is the standard relation used in the mathematical literature for the opposite of second-order stochastic dominance.

$x, y \in X$, it holds that

$$\mathcal{U}_i(x+y) \geq \mathcal{U}(x'+y') = \mathcal{U}_i(x') + \mathcal{U}_i(y') = \mathcal{U}_i(x) + \mathcal{U}_i(y)\,,$$

where x', y' have the same distribution as x, y, respectively, and are comonotonic. The last equality follows by law-invariance. It is easily seen that superadditivity extends to a minimum over Yaari functionals.

Finally, note that any utility functional that satisfies superadditivity and positive homogeneity also satisfies concavity[33] (and hence also diversification):

$$\mathcal{U}_i(\alpha x + (1-\alpha)y) \geq \mathcal{U}_i(\alpha x) + \mathcal{U}_i((1-\alpha)y) = \alpha\mathcal{U}_i(x) + (1-\alpha)\mathcal{U}_i(y)\,.$$

Proof for Proposition 2: For an incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{t})$ associated with equilibrium certainty equivalents $V_1(\cdot, \boldsymbol{q}, \boldsymbol{t}), \ldots, V_n(\cdot, \boldsymbol{q}, \boldsymbol{t})$, we define another mechanism $(\boldsymbol{q}, \boldsymbol{T})$ such that the allocation rule remains the same, while the transfers in the new mechanism $(\boldsymbol{q}, \boldsymbol{T})$ only depend on whether agents receive the object or not. Moreover, once the agent knows her type, her utility in $(\boldsymbol{q}, \boldsymbol{T})$ is a constant:

$$T_i(\boldsymbol{\theta}, r) = \begin{cases} \theta_i - V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t}) & \text{if } r \in W_i(\boldsymbol{\theta}) \\ -V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t}) & \text{else} \end{cases}.$$

By construction, the mechanism $(\boldsymbol{q}, \boldsymbol{T})$ is a *full-insurance mechanism.* Since $\mathcal{U}_i(b) = b$ for any $b \in \mathbb{R}$, if $(\boldsymbol{q}, \boldsymbol{T})$ is incentive compatible then it yields the same utility for each agent (given her type) as $(\boldsymbol{q}, \boldsymbol{t})$:

$$V_i(\theta_i, \boldsymbol{q}, \boldsymbol{T}) = V_i(\theta_i, \boldsymbol{q}, \boldsymbol{t}).$$

We show below that:

(a) $(\boldsymbol{q}, \boldsymbol{T})$ is incentive compatible and

(b) $(\boldsymbol{q}, \boldsymbol{T})$ is at least as profitable for the seller as $(\boldsymbol{q}, \boldsymbol{t})$.

(a) Note that a bidder's ex-post payoff from deviating in the original mechanism $(\boldsymbol{q}, \boldsymbol{t})$ and reporting θ_i' instead of θ_i equals the sum of two lotteries: the lottery obtained by the type θ_i' agent who reports truthfully - this is here a constant - and the lottery $\mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}}(\theta_i - \theta_i')$. Incentive compatibility of $(\boldsymbol{q}, \boldsymbol{t})$ and superadditivity imply that,

[33] See also Föllmer and Schied [2011] who show that every pair of these axioms imply the third.

for any θ_i and θ_i',

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) \geq V_i(\theta_i, \theta_i'; \boldsymbol{q}, \boldsymbol{t}) \geq V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{t}) + \mathcal{U}_i \left(\mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} (\theta_i - \theta_i') \right)$$

By construction, the last inequality implies that

$$\begin{aligned} V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}) &\geq V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{T}) + \mathcal{U}_i \left(\mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} (\theta_i - \theta_i') \right) \\ &= \mathcal{U}_i \left(V_i(\theta_i'; \boldsymbol{q}, \boldsymbol{T}) + \mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} (\theta_i - \theta_i') \right) \\ &= \mathcal{U}_i \left(\mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} \theta_i' - T_i\left(\theta_i', \boldsymbol{\theta}_{-i}, r\right) + \mathbf{1}_{\{r \in W_i(\theta_i', \boldsymbol{\theta}_{-i})\}} (\theta_i - \theta_i') \right) = V_i(\theta_i, \theta_i'; \boldsymbol{q}, \boldsymbol{T}) \end{aligned}$$

where the second line follows from cash invariance and where the third line follows from the property that the utility of a constant equals that constant. Since the right-hand-side is exactly the value the agent obtains from deviating in the mechanism $(\boldsymbol{q}, \boldsymbol{T})$, we have thus shown that $(\boldsymbol{q}, \boldsymbol{T})$ is incentive compatible.

(b) As the ex-post payoff in the constructed mechanism $(\boldsymbol{q}, \boldsymbol{T})$ is independent of θ_{-i} and r, cash invariance implies that

$$V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T}) + \mathcal{U}_i \left(T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r) \right)$$

As $V_i(\theta_i; \boldsymbol{q}, \boldsymbol{t}) = V_i(\theta_i; \boldsymbol{q}, \boldsymbol{T})$ by construction, it must hold that

$$0 = \mathcal{U}_i \left(T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r) \right) .$$

By weak risk aversion, an upper bound on this value is given by the expectation of the random variable $T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r)$:

$$0 = \mathcal{U}_i\big(T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r) \big) \leq \mathbb{E}[T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r) \mid \theta_i] ,$$

which further implies

$$\mathbb{E}[T_i(\boldsymbol{\theta}, r) - t_i(\boldsymbol{\theta}, r) \mid \theta_i] \geq 0.$$

We have thus established that the new mechanism $(\boldsymbol{q}, \boldsymbol{T})$ leads to a weakly higher expected revenue than the original mechanism $(\boldsymbol{q}, \boldsymbol{t})$.

Proof for Proposition 3: Note that for all i, $g_i(p)$ is non-decreasing, satisfies $g_i(0) = 0$, $g_i(1) = 1$, and

$$g_i(p) = \mathcal{U}_i(x(1, p)) \leq p.$$

We thus obtain for any $Q_i(\theta_i)$ that:

$$1 - g_i(1 - Q_i(\theta_i)) \geq Q_i(\theta_i) \geq g_i(Q_i(\theta_i)).$$

Both inequalities hold with equality only if $g_i(p) = p$ (risk-neutrality) or if $Q_i(\theta_i) \in \{0, 1\}$.

(Necessity): Fix a full-insurance mechanism $(\boldsymbol{q}, \boldsymbol{T})$ that implements $\boldsymbol{Q}$. Let $V_i(\theta_i)$ denote the certainty equivalent that agent i of type θ_i obtains in that mechanism. With a slight abuse of notation, we also use $V_i(\theta_i)$ to denote the lottery which yields a safe payoff of $V_i(\theta_i) \in \mathbb{R}$.

Let $x_i(\varepsilon_i, Q_i(\theta_i'))$ denote the lottery that yields ε_i with probability $Q_i(\theta_i')$ and 0 otherwise. If agent i of type θ_i deviates and reports to be of type θ_i', he will obtain a lottery that is the sum of $V_i(\theta_i')$ and $x_i(\theta_i - \theta_i', Q_i(\theta_i'))$. As the mechanism is incentive compatible, we have that, for any i and any θ_i, $\theta_i' \in [0, 1]$,

$$\mathcal{U}_i\Big(V_i(\theta_i)\Big) \geq \mathcal{U}_i\Big(V_i(\theta_i') + x_i(\theta_i - \theta_i', Q_i(\theta_i'))\Big).$$

By cash invariance, the above formula implies

$$V_i(\theta_i) \geq V_i(\theta_i') + \mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))\,.$$

Assume first that $\theta_i > \theta_i'$. Subtracting $V_i(\theta_i')$ and dividing by $\theta_i - \theta_i'$ yields that

$$\frac{V_i(\theta_i) - V_i(\theta_i')}{\theta_i - \theta_i'} \geq \frac{\mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))}{\theta_i - \theta_i'}\,.$$

The continuity of $\mathcal{U}_i$ together with homogeneity yields

$$\begin{aligned} \limsup_{\theta_i' \nearrow \theta_i} \frac{V_i(\theta_i) - V_i(\theta_i')}{\theta_i - \theta_i'} &\geq \limsup_{\theta_i' \nearrow \theta_i} \frac{\mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))}{\theta_i - \theta_i'} = \mathcal{U}_i(x_i(1, Q_i(\theta_i))) \\ &= g_i(Q_i(\theta_i)). \end{aligned} \tag{2}$$

Similarly, for any $\theta_i < \theta_i'$, we obtain that

$$\frac{V_i(\theta_i) - V_i(\theta_i')}{\theta_i - \theta_i'} \leq \frac{\mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))}{\theta_i - \theta_i'}\,.$$

which implies

$$
\begin{aligned}
\liminf_{\theta_i' \searrow \theta_i} \frac{V_i(\theta_i) - V_i(\theta_i')}{\theta_i - \theta_i'} &\leq \liminf_{\theta_i' \searrow \theta_i} \frac{\mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))}{\theta_i - \theta_i'} \\
&= -\mathcal{U}_i(x_i(-1, Q_i(\theta_i))) \\
&= 1 - \mathcal{U}_i(x_i(1, 1 - Q_i(\theta_i))) \\
&= 1 - g_i(1 - Q_i(\theta_i)) = 1 - g_i(1 - Q_i(\theta_i)) \qquad (3)
\end{aligned}
$$

where the first equality follows from positive homogeneity and the second equality follows from cash invariance: $1 + \mathcal{U}_i(x_i(-1, Q_i(\theta_i))) = \mathcal{U}_i(x_i(1, 1 - Q_i(\theta_i)))$.

Combining 2 and 3 yields that V is Lipschitz continuous and thus almost everywhere differentiable.

(Sufficiency) Consider now a reduced form allocation rule $\boldsymbol{Q}$ for which each component is non-decreasing and a mechanism such that

$$
g_i(Q_i(\theta_i)) \leq V_i'(\theta) \leq 1 - g_i(1 - Q_i(\theta_i)).
$$

The value of type θ_i of deviating and reporting type $\theta_i' < \theta_i$ is given by

$$
\begin{aligned}
V_i(\theta_i') + \mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))) &= V_i(\theta_i') + g_i(Q_i(\theta_i'))(\theta_i - \theta_i') \\
&= V_i(\theta_i) - \int_{\theta_i'}^{\theta_i} \left[V_i'(z) - g_i(Q_i(\theta_i'))\right] dz \\
&\leq V_i(\theta_i) - \int_{\theta_i'}^{\theta_i} \left[V_i'(z) - g_i\left(\limsup_{s \nearrow z} Q_i(s)\right)\right] dz \leq V_i(\theta_i)\,.
\end{aligned}
$$

Here, the first inequality follows from the assumption that Q_i is non-decreasing and $\theta_i' \leq z$. The second inequality follows from the assumption on V_i' and establishes that the agent does not want to deviate by under-reporting his value.

It is left to show that the agent cannot profitable deviate by reporting a value $\theta_i' > \theta_i$.

Over-reporting yields

$$\begin{aligned}
V_i(\theta_i') + \mathcal{U}_i(x_i(\theta_i - \theta_i', Q_i(\theta_i'))) &= V_i(\theta_i') - (\theta_i' - \theta_i)\left[1 - g_i\left(1 - Q_i(\theta_i')\right)\right] \\
&= V_i(\theta_i) + \int_{\theta_i}^{\theta_i'} \left[V_i'(z) - (1 - g_i(1 - Q(\theta_i')))\right] dz \\
&\leq V_i(\theta_i) + \int_{\theta_i}^{\theta_i'} \left[V_i'(z) - \left(1 - g_i(1 - \liminf_{s \searrow z} Q(s))\right)\right] dz \\
&\leq V_i(\theta_i)\,.
\end{aligned}$$

This completes the proof.

Proof for Corollary 1 1. As both $g_i(Q_i(\theta_i)$ and $1 - g_i(1 - Q_i(\theta_i))$ are bounded and thus Riemann integrable, Proposition 3 implies

$$V_i(0) + \int_0^{\theta_i} g_i(Q_i(t)dt \leq V_i(\theta_i) \leq V_i(0) + \int_0^{\theta_i} \left[1 - g_i(1 - Q_i(\theta_i))\right] dt.$$

In order to derive the upper bound on the revenue, note first that, in a full insurance mechanism, the expected revenue raised from agent i of type θ_i is given by

$$T_i(\theta_i) = Q_i(\theta_i)\theta_i - V_i(\theta_i)\,.$$

Thus, total revenue satisfies

$$\begin{aligned}
\sum_{i=1}^{n} \int_0^1 T_i(\theta_i) f_i(\theta_i) d\theta_i &= \sum_{i=1}^{n} \int \left[Q_i(\theta_i)\theta_i - V_i(\theta_i)\right] f_i(\theta_i) d\theta_i \\
&\leq \sum_{i=1}^{n} \int_0^1 \left[Q_i(\theta_i)\theta_i - V_i(0) - \int_0^{\theta_i} g_i(Q_i(t))dt\right] f_i(\theta_i) d\theta_i \\
&= \sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} g_i(Q_i(\theta_i))\right] f_i(\theta_i)\, d\theta_i - \sum_{i=1,2\ldots n} V_i(0)
\end{aligned}$$

where we used integration by parts, $g_i(0) = 0$ and $g_i(1) = 1$ to obtain the last equality.

Analogously, the lower bound for the total revenue is given by

$$\sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \left[1 - g_i(1 - Q_i(\theta_i))\right]\right] f_i(\theta_i)\, d\theta_i - \sum_{i=1}^{n} V_i(0)$$

2. The second part directly follows from the first part together with Proposition 2.

Proof for Proposition 4: Fix any full-insurance mechanism $(\boldsymbol{q}, \boldsymbol{T})$ that implements $\boldsymbol{Q}$. Let $V_i(\theta_i)$ denote the certainty equivalent that agent i of type θ_i obtains in that mechanism. Let $x(\theta_i - \theta_i', Q_i(\theta_i'))$ be the lottery that yields $\theta_i - \theta_i'$ with probability $Q_i(\theta')$ and 0 otherwise; $l(\theta_i - \theta_i', Q_i(\theta_i')) \equiv x(\theta_i - \theta_i', Q_i(\theta_i')) - Q(\theta_i')(\theta_i - \theta_i')$ is a mean-zero lottery. As the mechanism is incentive compatible we have that for all $\theta_i' < \theta_i$,

$$V_i(\theta_i) \geq V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i') + l(\theta_i - \theta_i', Q_i(\theta_i'))\,.$$

Note that in the above formula, $V_i(\theta_i)$ and $V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i')$ are both constants.

For any constant b and any lottery l, let $\pi_i(b, l)$ be the risk premium assign to the combined lottery $b + l$. By definition of the risk premium the above equation is equivalent to

$$\begin{aligned}
&\tilde{\mathcal{U}}_i\Big(V_i(\theta_i)\Big) \geq \tilde{\mathcal{U}}_i\Big(V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i') - \pi(V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i'), l(\theta_i - \theta_i', Q_i(\theta_i'))\Big) \\
\iff\ & V_i(\theta_i) \geq V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i') - \pi(V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i'), l(\theta_i - \theta_i', Q_i(\theta_i'))\,.
\end{aligned}$$

Subtracting $V(\theta_i')$ and dividing by $(\theta_i - \theta_i')$ yields that

$$\frac{V_i(\theta_i) - V_i(\theta_i')}{\theta_i - \theta_i'} \geq Q_i(\theta_i') - \frac{1}{\theta_i - \theta_i'}\pi\Big(V_i(\theta_i') + Q_i(\theta_i')(\theta_i - \theta_i'), l(\theta_i - \theta_i', Q_i(\theta_i'))\Big)\,.$$

Taking the lim sup on both sides yields

$$\begin{aligned}
\limsup_{\theta_i' \nearrow \theta_i} \frac{V_i(\theta_i) - V_i(\theta_i')}{\theta_i - \theta_i'} &\geq \limsup_{\theta_i' \nearrow \theta_i} Q_i(\theta_i') - \limsup_{\theta_i' \nearrow \theta_i} \frac{1}{\theta_i - \theta_i'}\pi\Big(V_i(\theta_i), l(\theta_i - \theta_i', Q_i(\theta_i'))\Big) \\
&= Q_i(\theta_i) - \limsup_{\theta_i' \nearrow \theta_i} \frac{1}{\theta_i - \theta_i'}\pi\Big(V_i(\theta_i), l(\theta_i - \theta_i', Q_i(\theta_i'))\Big) \\
&= Q_i(\theta_i)
\end{aligned}$$

The last equality follows from the definition of second-order risk aversion. As $Q_i(\theta_i)$ is bounded and Riemann integrable, the above implies

$$V_i(\theta_i) \leq V_i(0) + \int_0^{\theta_i} Q_i(t)dt$$

The above inequality further implies that the seller's expected revenue satisfies

$$\tilde{R}(\boldsymbol{Q}) \leq \sum_{i=1}^{n} \int_0^1 \left[\theta_i Q_i(\theta_i) - Q_i(\theta_i) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] f_i(\theta_i) \, d\theta_i - \sum_{i=1,2\ldots n} V_i(0) \, .$$

as desired.

Proof of Theorem 1: **1**. For every positive measure λ on $[0,1]$ and every feasible Q we have that

$$\int_0^1 \int_\theta^1 [Q(s) - F^{n-1}(s)] f(s) ds \, d\lambda(\theta) \leq 0 \, .$$

Changing the order of integration yields that the above condition is equivalent to

$$\int_0^1 \lambda([0,s]) \big(Q(s) - F^{n-1}(s) \big) f(s) ds \leq 0 \, .$$

Consequently, for every λ and every feasible Q , an upper bound on the objective function is given by

$$\int_0^1 \left[\theta Q(\theta) - g(Q(\theta)) \frac{1 - F(\theta)}{f(\theta)} - \lambda([0,\theta]) \big(Q(\theta) - F^{n-1}(\theta) \big) \right] f(\theta) d\theta \, . \tag{4}$$

Setting $\lambda([0,\theta]) = \max\{\theta - g'(F^{n-1}(\theta))[1 - F(\theta)]/f(\theta), 0\}$ the above value simplifies to

$$\begin{aligned} &\int_0^{\theta^*} \left[\theta Q(\theta) - g(Q(\theta)) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta \\ &\quad + \int_{\theta^*}^1 \left[Q(\theta) g'(F^{n-1}(\theta)) - g(Q(\theta)) \right] (1 - F(\theta)) \, d\theta + \int_{\theta^*}^1 \lambda([0,\theta]) F^{n-1}(\theta) d\theta \, . \end{aligned} \tag{5}$$

As g is convex, it lies at almost every point above its tangent, and we obtain for almost every $\theta \in [\theta^*, 1]$ and all $q \in [0,1]$

$$g(F^{n-1}(\theta)) + g'(F^{n-1}(\theta))(q - F^{n-1}(\theta)) \leq g(q) \, .$$

This is equivalent to

$$q \, g'(Q^*(\theta)) - g(q) \leq Q^*(\theta) \, g'(Q^*(\theta)) - g(Q^*(\theta)) \, ,$$

for almost every $\theta \in [\theta^*, 1]$ and all $q \in [0,1]$. Thus Q^* is a pointwise maximizer of the relaxed problem (5). As $Q^*(\theta) = F^{n-1}(\theta)$ for $\theta \in [\theta^*, 1]$ and $\lambda([0,\theta]) = 0$ for $\theta \in [0, \theta^*]$ we have that

$$\int_0^1 \lambda([0,\theta])\big(Q(\theta) - F^{n-1}(\theta)\big)f(\theta)d\theta = 0$$

Thus, the upper bound on the virtual value given in (4) is achieved by (λ, Q^*). Denote by $\mathcal{D} = \{\hat{Q} : [0,1] \to [0,1]$ such that $\int_\theta^1 \hat{Q}(s)f(s)ds \leq \int_\theta^1 F^{n-1}(s)f(s)ds$ for all $\theta \in [0,1]\}$. As $Q^* \in \mathcal{D}$ we obtain

$$\begin{aligned}
&\max_{Q\in\mathcal{D}} \int_0^1 \left[\theta Q(\theta) - g(Q(\theta))\frac{1-F(\theta)}{f(\theta)}\right] f(\theta)d\theta \\
&\quad\leq \max_{Q\in\mathcal{D}} \int_0^1 \left[\theta Q(\theta) - g(Q(\theta))\frac{1-F(\theta)}{f(\theta)} - \lambda([0,\theta])\big(Q(\theta) - F^{n-1}(\theta)\big)\right] f(\theta)d\theta \\
&\quad= \int_0^1 \left[\theta Q^*(\theta) - g(Q^*(\theta))\frac{1-F(\theta)}{f(\theta)} - \lambda([0,\theta])\big(Q^*(\theta) - F^{n-1}(\theta)\big)\right] f(\theta)d\theta \\
&\quad= \int_0^1 \left[\theta Q^*(\theta) - g(Q^*(\theta))\frac{1-F(\theta)}{f(\theta)}\right] f(\theta)d\theta\,.
\end{aligned}$$

This establishes that Q^* is optimal if it can be implemented in an incentive compatible full-insurance mechanism. As shown in Proposition 3 a sufficient condition is that Q^* is non-decreasing. As g is convex, g' and g'^{-1} are non-decreasing, and thus Q^* is non-decreasing. This completes the proof of part 1.

2. Let Q_n be the optimal reduced form allocation with n bidders, $i \in \{1, ..., n\}$. Assume now that there are $n+1$ bidders, $i \in \{1, ..., n, n+1\}$. Using the allocation Q_n for bidders $1, ..., i-1, i+1, ..., n+1$ while completely excluding bidder i is a feasible (i.e., incentive compatible and individually rational mechanism) in the $n+1$ bidder problem. Each of these proposed asymmetric allocations yields the same revenue as the optimal mechanism for the n bidder problem. Averaging these $n+1$ asymmetric allocations yields a symmetric, feasible allocation for the $n+1$ bidder problem. By the concavity of the revenue, the symmetric average allocation yields a higher revenue than each asymmetric one. In particular, the optimal allocation for the $n+1$ bidder problem (that is known to be symmetric) yields at least as much revenue as the optimal allocation for the n bidder problem.

Proof for Lemma 2: Taking the derivative of the function

$$\theta - g'(F^{n-1}(\theta))\frac{1-F(\theta)}{f(\theta)}$$

yields

$$1 - g'(F^{n-1}(\theta))\left[\frac{1-F(\theta)}{f(\theta)}\right]' - (n-1)F^{n-2}(\theta)g''(F^{n-1}(\theta))[1-F(\theta)]$$

As we have assumed a monotone hazard rate, the second terms is always non-negative. We now show that, for n large enough,

$$(n-1)F^{n-2}(\theta)g''(F^{n-1}(\theta))[1-F(\theta)] \leq 1$$

holds for any $\theta \in [0,1]$, which implies the result. Note that, over $p \in [0,1]$,

$$(n-1)p^{n-2}(1-p)$$

is maximized at $p = \frac{n-2}{n-1}$. Plugging in $p = \frac{n-2}{n-1}$ yields that

$$\max_{p\in[0,1]}(n-1)p^{n-2}(1-p) \leq \left(\frac{n-2}{n-1}\right)^{n-2}.$$

Hence

$$\begin{aligned} &\lim_{n\to\infty}\{(n-1)F^{n-2}(\theta)g''(F^{n-1}(\theta))[1-F(\theta)]\} \\ < \;& e\cdot\lim_{n\to\infty}\left(\frac{n-2}{n-1}\right)^{n-2} = 1 \end{aligned}$$

as desired.

Proof of Theorem 2: **1.** The admissible set $\{x(t),\ u(t)\}$ is non-empty, convex and compact, and the functions that appear in the integrand and in the constraints, namely $tf(t)u - g_{\min}(u)(1-F(t))$, $uf(t)$, and $\int_t^1 F^{n-1}(z)f(z)dz + x$, are continuous in their respective variables t, u, x. Existence follows by Theorem 5.5 in SS [1987].

2. This folllows by the above continuity condition and by the observation that the set of monotonic functions that satisfy the feasibility constraint 3 is convex and also compact in the L_1 strong topology (see Kleiner et al [2021]).

3. Recall that the Hamiltonian H is independent of the state x, and that the feasibility constraint $\int_t^1 F^{n-1}(z)f(z)dz + x \geq 0$ is linear in x. Sufficiency follows then by Theorem 5.1 in SS [1987].

Proof for Lemma 3: By Corollary 1 the expected revenue in the optimal mechanism is given by

$$R^{j*} = n \int_0^1 \left[\theta Q^{j*}(\theta) - \frac{1 - F(\theta)}{f(\theta)} g^1(Q^{j*}(\theta)) \right] f(\theta)\, d\theta,$$

where Q^{j*} is the optimal allocation when utilities are represented by set Φ^j, $j = 1, 2$. We obtain the following inequalities

$$\begin{aligned} R^{2*} &= n \int_0^1 \left[\theta Q^{2*}(\theta) - \frac{1 - F(\theta)}{f(\theta)} g^2(Q^{2*}(\theta)) \right] f(\theta)\, d\theta \\ &\leq n \int_0^1 \left[\theta Q^{2*}(\theta) - \frac{1 - F(\theta)}{f(\theta)} g^1(Q^{2*}(\theta)) \right] f(\theta)\, d\theta \\ &\leq n \int_0^1 \left[\theta Q^{1*}(\theta) - \frac{1 - F(\theta)}{f(\theta)} g^1(Q^{1*}(\theta)) \right] f(\theta)\, d\theta = R^{1*}. \end{aligned}$$

The first inequality follows because the allocation Q^{2*} is monotone (and hence implementable also if the preferences are represented by Φ^1 and by the comparison of risk aversions). The second inequality follows because adjusting the allocation rule to be optimal given that preferences that are represented by Φ^1 further increases the expected revenue.

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