The Optimal Allocation of Prizes in Contests

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Abstract: We study a contest with multiple (not necessarily equal) prizes. Ex-ante symmetric, risk-neutral contestants have independently distributed private information about an ability parameter that affects their costs of bidding. The contestant with the highest bid wins the first prize, the contestant with the second-highest bid wins the second prize, and so on until all the prizes are allocated. All contestants incur their respective costs of bidding. The contest's designer maximizes the expected sum of bids. Our main results are: 1) We display symmetric bidding equilibria for contestants with linear, convex or concave cost functions. 2) If the cost functions are linear or concave, then, it is optimal for the designer to allocate the entire prize sum to a single "first" prize. 3) We give a necessary and sufficient condition ensuring that several prizes are optimal if contestants have convex cost functions. 4) Even if the designer can use instruments that exclude types with relatively low ability (whose increased bidding cause the benefit of having more prizes), the award of several prizes is advantageous.

(JEL: D44, J31, D72, D82)

1. Introduction

In 1902 Francis Galton posed the following problem:

> "A certain sum, say £100, is available for two prizes to be awarded at a forthcoming competition; the larger one for the first of the competitors, the smaller one for the second. How should the £100 be most suitably divided between the two ? What ratio should a first prize bear to that of a second one ? Does it depend on the number of competitors, and if so, why ?" Galton (*Biometrika*, Vol.1, 1902)

In his article Galton proposes a ratio of 3 to 1 to the above question. Since Galton does not explicitly state what is the contest designer's goal, his answer is somewhat arbitrary. Nevertheless, his work is important for it pioneered both the scientific literature on contests and the use of the so called *order statistics*[1].

Contests are situations in which agents spend resources in order to win one or more prizes. A main feature is that, independently of success, all contestants bear some costs. Many multiple-prize contests arise naturally[2], while others are designed in order to achieve specific goals. The prevalence of multiple-prize contests is obvious: employees spend effort in order to be promoted in organizational hierarchies, which often consist of several types of well-defined positions; students compete for grades in exams[3]; in procurement contests runners-up often serve as "second sources"; in proportional parliamentary systems politicians compete for ranked places on the party's list[4]; athletes compete for gold, silver and bronze medals, or for monetary prizes; architectural competitions for prominent structures attract several designs by offering substantial monetary prizes[5]; young pianists compete for the first, second and third prizes in the Rubinstein international competition, etc...Of great interest are also technological inducement prizes offered to individuals or groups who provide best entries in a contest, or

[1] 3 is the limit when the number of contestants (who have normally distributed abilities) goes to infinity of the ratio between 1) the expected difference between the value of the highest order statistic and the second highest, and 2) the expected difference between the value of the highest order statistic and the third highest statistic. For relations involving order statistics and their applications see Arnold and Balakrishnan (1989) and Shaked and Shantikumar (1994).

[2] e.g., animals compete for territories or mates of different qualities.

[3] At least in the U.S. many professors grade on a curve that allocates a fixed percentage of A's, B's, C's, D's and F's.

[4] in the U.S., the vice-president job is often considered to be a consolation prize.

[5] For example, designers for the German pavillion at the EXPO 2000 in Hannover competed over 7 prizes of DM 161000, DM 119000, DM 91000, DM 63000, DM 49000, DM 42000, DM 35000. In this kind of competitions, the actual designer need not be one of the prize winners.

who first meet some specified technical goal. Here are a few examples for the "best entries" type[6]: The European Information Technology Society annually awards three grand prizes worth 200000 euros each[7] for "novel products with high information technologies content and evident market potential"; The FCC's Pioneer Preference Program offered guaranteed slices of spectrum to companies that developed and implemented innovative communication services and technologies[8]; The privately funded Loebner prize is annually awarded to the computer program that is the most "human" in its responses to inquiries.

In the technological examples above, the designer's goal is not to achieve the highest possible top performance (or some pre-specified level of that performance), but rather to induce a general increase of activity in the specific field[9]. Similarly, professors wish to maximize the expected learning effort made by their students[10], organizers of athletic or artistic competitions often need to maximize average

[6] The most famous example for the "specified technical goal" type is surely the British Longitude Act, issued in 1714, which specified three prizes (£ 20000, £15000, £10000; these are equivalents of millions of today's dollars) for methods to determine longitude in varying degrees of accuracy (see Sobel, 1995). For more recent examples, such as the Feynman Prize for technical progress in nanotechnology, or the super efficient refrigerator prize, see Windham (1999).

[7] There are also 20 prizes worth 5000 euros each.

[8] Three companies were offered pioneer status in 1992. The program probably made sense in an era where spectrum allocation was done by lottery or bureaucratic process. Congress terminated the programme in 1997, after the advent of spectrum auctions.

[9] Another well known example of this type are the early aviation prizes, offered to stimulate the fledgling aeronautic industry. Between the first flight by the Wright brothers and 1929, over 50 major prizes were offered by governments, individuals and corporations. In 1926-7 alone, Daniel Guggengeim offered more than $ 2.5 million in prizes.

[10] Imagine an exam where it is announced that the top student will get an A while the rest will fail. It seems obvious that most students will not bother to learn at all, a rather undesirable outcome. A similar idea can be found behind the award of tenure to more than one assistant professor.

performance (or some related measure) in order to thrill audiences, and local authorities that organize gardening competitions wish to improve the community's appearance .

Most of the literature (see references below) has treated contests where a unique prize is awarded. Intuitively, the award of a single prize seems consistent with a general intuition about the efficiency of rewarding only the best (and supposedly ablest) competitor[11]. But, given the wealth of real-life multi-prize examples, it is of interest to offer a rationale for both winner-take-all and multiple-prize contests in a single, integrated model. Specifically, we address Galton's problem in the following framework: Several risk-neutral agents engage in a contest where multiple prizes with known and common values are awarded. Each contestant i submits a bid (or undertakes an observable "effort"). The contestant with the highest bid wins the first prize, the contestant with the second-highest bid wins the second prize, and so on until all the prizes are allocated. All contestants (including those that did not win any prize) incur a cost that is a strictly increasing function of their bid. This function is common knowledge. We differentiate among the cases where the cost function is, respectively, linear, concave or convex in effort. The cost function of contestant i also depends on a parameter (say "ability") that is *private information* to that player. The main assumption we make is one of separability between ability and bid in the cost function. The function governing the distribution of abilities in the population is common knowledge, an abilities are drawn independently of each other[12]. Each contestant chooses

[11] So called "winner-take-all" contests are the subject of an entertaining book by Frank and Cook (1995), but many of their examples are in fact multi-prize contests.

[12] It seems reasonable to assume the existence of a "natural" distribution of abilities, out of which drwas are taken. The independence assumption is problematic in some models with endogenous entry. Both assumptions were postulated by Galton who considered independent draws from the normal distribution, and they allow us to explicitly compute symmetric equilibria

his bid in order to maximize expected utility (given the other competitors' bids and given the values of the different prizes). The goal of the contest designer is to maximize the total expected effort (i.e., the expected sum of the bids) at the contest[13]. The designer can determine the number of prizes having positive value and the distribution of the fixed total prize sum among the different prizes.

Given the above assumptions, we display symmetric bidding equilibria for any number of prizes and contestants with linear, concave or convex cost functions. In order to have a less technical exposition, we focus however on the designer's problem in the case where she can award two (potentially unequal) prizes, and where there are at least three contestants. Compared to one-prize contests, this case already displays the main ingredient for complexity in bidding[14]. It will become clear that none of our qualitative results changes if we allow for more than two prizes.

In the framework of our model, we can answer Galton's questions as follows: 1) If the contestants having linear or concave cost functions it is optimal for the designer to allocate the entire prize sum to a single "first" prize. 2) We give a

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by analyzing one differential equation instead of a complex system.

[13]In some situations that fit our model, the designer has other goals. For example, in a lobbying model the contest designer might not be the beneficiary of the "wasteful" lobbying activities, and she might wish to minimize them. Our analysis can be easily extended to other goal functions since we explicitly display bidding equilibria (that are independent of the designer's goal).

[14]If there is only one prize, or if there are several equal prizes, each contestant perceives two payoff-relevant alternatives: I win a prize, or I win nothing. Hence, bids are determined by the difference in expected payoff between those two alternatives (the same logic applies if there are two unequal prizes but only two contestants). If there are at least two unequal prizes and at least three contestants, each contestant perceives at least three payoff relevant alternatives (I win the first prize, I win the second prize,..., I win nothing). Bids are now determined by several differences in expected payoffs.

necessary and sufficient conditions ensuring that (at least) two prizes are optimal if the contestants have convex cost functions. Depending on the parameters, the optimal prize structure may involve then several equal prizes or different prizes whose ratio can be computed. 3) If the contestants have convex cost functions, several prizes may be optimal even if the contest designer can use instruments (such as entry fees or minimum bid requirements) that exclude types with relatively low abilities.

We now briefly describe the intuition behind our results. The equilibrium bid depends on the contestant's cost (and hence on his ability), on the probabilities of winning different prizes, and on the prizes' values[15]. Moreover, the equilibrium bid is increasing in ability. Since a player with higher ability has a higher chance to win the first prize, increasing the value of the first prize by one penny causes an overall increase in equilibrium bids (and the increase is higher for higher abilities). In contrast, the probability of getting the second prize is not monotone in ability[16], and therefore the marginal effect of the second-prize (and of all other prizes but the first) is ambiguous. The marginal effect of the second prize on the equilibrium bid function is in fact negative for players with high enough abilities, but it is positive for middle and low ability players. Moreover, for contestants with abilities below a certain threshold, the (positive) marginal effect of the second prize is higher than the marginal effect of the first prize, since these types are more likely to get the second prize rather than the first. The relevant variable for a designer who wants to maximize the average (i.e., expected) bid of each contestant becomes the average difference between the marginal effects of the second prize and the first

[15] If there are p prizes, the bid function involves linear combinations of p order-statistics. The equilibrium bid for contestants with concave or convex cost functions is obtained by applying the inverse of the cost function to the equilibrium bid for linear cost functions.

[16] For example, both the lowest and highest ability types have a zero probability of getting the second prize if there are at least three contestants.

prize, respectively. If the average difference is negative the designer should award only a unique (first) prize - this turns out to be the case for contestants with linear or concave cost functions, no matter what the (common) distribution of abilities is[17]. The opposite can happen only for convex cost functions, where the positive effect of further prizes on "middle ability" types can more than compensate the decreased bidding of the ablest competitors. Two or more prizes may be optimal in this case. The optimal allocation of the prize sum among the several prizes depends then on the number of contestants, the distribution of abilities, and on the effort cost function. Finally, we show by way of example that, for convex cost functions, the beneficial effect of additional prizes persists even if types of relatively low ability can be excluded by instruments as entry fees or minimum bid requirements.

The paper is organized as follows: In Section 2 we present the contest model with multiple prizes and private information about a parameter (e.g. ability) entering cost functions. In Section 3 we focus on linear cost functions and we first derive the symmetric equilibrium bid functions. Then we formulate the contest designer's problem and we prove that it is optimal to award a single prize. In Section 4 we use the result obtained above in order to study the optimal prize structure for contestants with concave and convex cost functions. We illustrate the non-trivial optimal prize structure in an example with convex cost functions. In Section 5 we briefly study the effects of entry fees. In Section 6 we gather concluding comments. All proofs appear in an Appendix.

[17] A-priori it seems that the sign of the average difference will depend on features of the distribution of abilities, such as, say, the relative weight of "middle" types.

1.1. Related Literature

The economic literature on contests is very large, and most (but not all) of it has focused on the case of one prize[18]. Contest models with complete information about the value of a unique prize include, among others: Tullock (1980), Varian (1980)[19], Moulin (1986), Dasgupta (1986), Hillman and Samet (1987), Dixit (1987), Snyder (1989), Ellingsen (1991), Baye et. al. (1993), Baye et. al. (1996). The last paper offers a complete characterization of equilibrium behavior in the complete information all-pay auction with one prize[20]. All-pay auction models with incomplete information about the prize's value to different contestants include Weber (1985), Hillman and Riley (1989), Amann and Leininger (1996), and Krishna and Morgan (1997). Equilibrium uniqueness in such models with two players is treated in Amann and Leininger (1996) and Lizzeri and Persico (2000).

Research tournaments models with one prize are discussed in Wright (1983), Taylor (1995) and Fullerton and McAfee (1999).

The use of contests in order to extract effort under "moral hazard" conditions has been first emphasized by Lazear and Rosen (1981). Their work has been extended in many directions, by, among others, Green and Stokey (1983) , Nalebuff and Stiglitz (1983), and Rosen (1986)[21]. A common assumption in these papers is that the observed output is a stochastic function of the unobservable effort. All

[18] Applications have been made to rent-seeking, lobbying, technological races, political contests, promotions in labor markets, trade wars, military and biological wars of attrition, etc...

[19] Varian's is a model of sales, in fact a mirror-image of an all-pay auction. The bidding direction is reversed, as the firm setting the lowest sale price gets the extra demand of informed customers.

[20] Contrary to earlier erroneous claims, there are many equlibria.

[21] Ehrenberg and Bognanno (1990) and Knoeber and Thurman (1994) test several predictions of this body of theory using observed prize structures in professional golf tournaments, and reward schemes for broiler producers, respectively.

agents have the same known ability. Lazear and Rosen derive the optimal prize structure in a contest with two workers and two prizes and compare it to optimal piece rates. Krishna and Morgan (1998) assume that the contest designer has a fixed prize purse and study the optimal allocation of the purse among several non-negative prizes in contests with 2, 3 or 4 contestants.

Our model is isomorphic to a "private values" all-pay auction with several (potentially unequal) prizes. Broecker's (1990) model of credit markets has several features of an all-pay auction with as many prizes as contestants[22]. Wilson (1979) and Anton and Yao (1992) study split award auctions that can be also interpreted as contests with several prizes[23]. Clark and Riis (1998) study contests with multiple identical prizes under complete information and compare simultaneous versus sequential designs from the point of view of a revenue-maximizing designer. Bulow and Klemperer (1999) study an incomplete information model of a war of attrition with K identical prizes and $N + K$ contestants.

Barut and Kovenock (1998) study a complete information multi-prize contest with heterogenous prizes[24]. In this symmetric, complete information environment, they show that the revenue maximizing prize structure allows any combination of $K - 1$ prizes, where K is the number of contestants. (In particular, allocating the entire prize sum to a unique first prize is optimal.)

Our paper is close in focus to Glazer and Hassin (1988). Besides studying the symmetric equilibria of a multi-prize complete information model, these authors also propose an incomplete information model that is more general than ours since it allows for both cost functions that are not necessarily separable in ability

[22] With only two banks his model is isomorphic to Varian's (1980) model of sales.

[23] Anto and Yao also mention examples such as split home schedules of professional sport franchises wishing to maximize amenities offered by several municipalities.

[24] In our terminology, players have linear cost functions and the same, common-knowledge ability.

and bid and for risk averse contestants. But this general structure is not easily amenable to analysis, and, in order to obtain results, the authors further assume a separable and linear cost function such that the lowest ability type has an infinite cost of bidding, a uniform ability distribution, risk neutral contestants, and no entry fees. With these assumptions, they show that a unique first prize is optimal.

2. The Model

Consider a contest where p prizes are awarded. The value of the $j-th$ prize is V_j, where $V_1 \geq V_2 \geq ... \geq V_p \geq 0$. The values of the prizes are common knowledge. We assume that $\sum_{i=1}^{p} V_i = 1$ - this is just a normalization.

The set of contestants is $K = \{1, 2, ..., k\}$. Without loss of generality we can assume that $k \geq p$ (i.e., there are at least as many contestants as there are prizes).

At the contest each player i makes a bid x_i. Bids are submitted simultaneously. A bid x_i causes a disutility (or cost) denoted by $c_i\gamma(x_i)$, where $\gamma : R_+ \to R_+$ is a strictly increasing function with $\gamma(0) = 0$, and where $c_i > 0$ is an ability parameter[25]. Note that a **low** c_i means that i has a **high** ability (i.e., lower cost) and vice-versa.

The ability (or *type*) of contestant i is private information to i. Abilities are drawn independently of each other from an interval $[m, 1]$ according to the distribution function F which is common knowledge. We assume that F has a continuous density $F' > 0$. In order to avoid infinite bids caused by zero costs, we assume that m, the type with highest possible ability, is strictly positive[26].

[25] The treatment of the case in which $i's$ cost function is given by $\delta(c_i)\gamma(x_i)$, where δ is strictly monotone increasing, is completely analogous. The main assumption here is the separability of ability and bid.

[26] The case where $m = 0$ can be treated as well, but requires slightly different methods. The choice of the interval $[m, 1]$ is a normalization.

The contestant with the highest bid wins the first prize V_1. The contestant with the second highest bid wins the second prize V_2, and so on until all the prizes are allocated[27]. That is, the payoff of contestant i who has ability c_i , and submits a bid x_i is either $V_j - c_i\gamma(x_i)$ if i wins prize j, or $-c_i\gamma(x_i)$ if i does not win a prize.

Each contestant i chooses his bid in order to maximize expected utility (given the other competitors' bids and the values of the different prizes). The contest designer determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximize the expected value of the sum of the bids $\sum_{i=1}^{k} x_i$ (given the contestants' equilibrium bid functions).

3. Linear Cost Functions

We assume here that the cost functions are linear, i.e., $\gamma(x) = x$. The next Proposition displays the symmetric equilibrium bid with two prizes and $k \geq 3$ contestants[28]. In the Appendix we also provide the general formula for the equilibrium bid functions with $p > 2$ prizes.

Proposition 3.1. *Assume that there are 2 prizes, $V_1 \geq V_2 \geq 0$, and $k \geq 3$ contestants. In a symmetric equilibrium[29], the bid function of each contestant is*

[27]If $h > 1$ bids tie for a prize, each respective bidder gets the prize with probability $\frac{1}{h}$.

[28]As mentioned in the introduction, if there are only two contestants, the situation is isomorphic to the one where there is a unique prize whose value is equal to the difference between the two prizes. Hence, it is trivially true that awarding a unique prize is optimal for the contest's designer.

[29]It is not too difficult to show that this is the unique symmetric equilibrium. By analogy to the properties of incomplete information pay-your-bid auction models with a continuum of types, we conjecture that this remains the unique equilibrium even if asymmetric strategies are considered. But a proof of such a claim (or the construction of a counter-example) is likely to be very complex. The only known results show uniqueness for the two-bidders case (see Amman

given by $b(c) = A(c)V_1 + B(c)V_2$*, where:*

$$A(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-2}F'(a)da \tag{3.1}$$

$$B(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-3}[((k-1)F(a)-1]F^{'}(a)da \tag{3.2}$$

Proof. See Appendix ■

3.1. The Designer's Problem

Let $V_2 = \alpha$ and $V_1 = 1-\alpha$, where $0 \leq \alpha \leq \frac{1}{2}$ (since the second prize must be smaller than the first). By Proposition 3.1, each contestant's equilibrium bid function is given by

$$b(c) = (1-\alpha)A(c) + \alpha B(c) = A(c) + \alpha(B(c) - A(c)).$$

The average bid of each contestant is given by

$$\int_m^1 (A(c) + \alpha(B(c) - A(c)))F'(c)dc.$$

Since there are k contestants, the seller's problem is:

$$\max_{0\leq\alpha\leq\frac{1}{2}} k\int_m^1 (A(c) + \alpha(B(c) - A(c)))F'(c)dc$$

The above problem is equivalent to:

$$\max_{0\leq\alpha\leq\frac{1}{2}} \alpha\int_m^1 (B(c) - A(c)))F'(c)dc \tag{3.3}$$

The solution to Problem 3.3 is extremely simple: if the integral is positive, then the optimal α is $\frac{1}{2}$ (i.e., award two equal prizes). Otherwise, the optimal α

and Leininger, 1996 and Lizzeri and Persico, 1999) is zero (i.e., award a unique prize). A "common-sense" conjecture is that the sign of the integral will depend on the specific properties of the distribution function. But Lemma 8.1 in the Appendix shows that the integral in Problem 3.3 is always negative. Hence the solution to Problem 3.3 must be $\alpha = 0$[30]. Thus we obtain:

Proposition 3.2. *Assume that the designer can award at most two prizes,* $V_1 \geq V_2 \geq 0$, *and that there are* $k \geq 3$ *contestants with linear cost functions. Then it is optimal to allocate the entire prize sum to a single first prize.*

It is important to note that the above result holds even if, a-priori, the seller is allowed to award more than two prizes (for the argument, see footnote 41 in the Appendix.)

The following example illustrates Proposition 3.2. In particular, we give explicit formulas for the equilibrium bid functions when the distribution of abilities is uniform.

Example 3.3. *Assume that* $F(c) = \frac{1}{1-m}c - \frac{m}{1-m}$, *i.e., abilities are uniformly distributed on the interval* $[m, 1]$. *We obtain that:*

$$
\begin{aligned}
A(c) &= (\frac{1}{1-m})^{k-1}(1-k)(\sum_{s=1}^{k-2}\frac{(1-c)^s}{s} + \ln c) \\
B(c) &= (\frac{1}{1-m})^{k-1}(k-1)[\sum_{s=1}^{k-2}\frac{(1-c)^s}{s} + \ln c \\
&\quad +(1-c)^{k-2} + m(k-2)(\sum_{s=1}^{k-3}\frac{(1-c)^s}{s} + \ln c)]
\end{aligned}
$$

[30] As noted in the introduction, we can easily deal with cases where the designer has other goals. For example, if the designer wants to minimize the the expected sum of bids, it should award two equal prizes, i.e., $\alpha = \frac{1}{2}$ is optimal .

Assume now that $k = 3$, $m = \frac{1}{2}$ and $F(a) = 2a - 1$ (i.e., uniform distribution on the interval $[\frac{1}{2}, 1]$). The formulas above yield:

$$\begin{aligned} A(c) &= -8 + 8c - 8\ln c \\ B(c) &= 16 - 16c + 12\ln c \\ \int_m^1 (B(c) - A(c))F'(c)dc &= 2\int_{\frac{1}{2}}^1 (24 - 24c + 20\ln c)dc \\ &= -14 + 20\ln 2 = -0.137 \end{aligned}$$

.

$A(c)$ - thick line ; $B(c)$ - thin line; $\frac{1}{2}(A(c) + B(c))$ - dotted line

Note that the curve $A(c)$ also describes the equilibrium bid when there is a unique price $V_1 = 1$. For comparison, we have also plotted the resulting equilibrium bid function when there are two equal prizes, $V_1 = V_2 = \frac{1}{2}$.

4. Concave and Convex Cost Functions

Assume now that a bidder with ability c has a cost function given by $c\gamma$ such that $\gamma(0) = 0$, $\gamma' > 0$. Let $g = \gamma^{-1}$, and observe that $g' > 0$.

Proposition 4.1. *Assume that there are 2 prizes, $V_1 \geq V_2 \geq 0$, and $k \geq 3$ contestants. In a symmetric equilibrium, the bid function of each contestant is given by $b(c) = g[A(c)V_1 + B(c)V_2]$, where $A(c)$ and $B(c)$ are defined by equations 3.1 and 3.2, respectively.*

Proof. See Appendix. ■

4.1. The Designer's Problem

Let $V_1 = 1 - \alpha$, and $V_2 = \alpha$, where $0 \leq \alpha \leq \frac{1}{2}$. Analogous to the case of linear cost functions, the designer's revenue with concave or convex cost functions is given by

$$R(\alpha) = k \int_m^1 g[A(c) + \alpha(B(c) - A(c))]F'(c)dc$$

and the designer's problem is given by

$$\max_{0 \leq \alpha \leq \frac{1}{2}} k \int_m^1 g[A(c) + \alpha(B(c) - A(c))]F'(c)dc$$

Roughly speaking, the main new effects are due to the fact that the beneficial marginal effect of the second-prize on middle and low ability players is amplified when contestants have convex cost functions, while the opposite occurs for concave cost functions. We obtain the following results:

Proposition 4.2. *Assume that the designer can award at most 2 prizes, $V_1 \geq V_2 \geq 0$, and that there are $k \geq 3$ contestants with concave cost functions. Then it is optimal to allocate the entire prize sum to a single first prize.*

Proof. See Appendix. ■

Proposition 4.3. *Assume that the designer can award at most 2 prizes, $V_1 \geq V_2 \geq 0$, and that there are $k \geq 3$ contestants with convex cost functions. A* *necessary and sufficient condition for the optimality of two prizes is given by*

$$\int_m^1 (B(c) - A(c))g'(A(c))F'(c)dc > 0 \tag{4.1}$$

If condition 4.1 is satisfied[31] *then, depending on the convexity of the cost function, it is either optimal to award two prizes* $V_1 = 1 - \alpha^*$ *and* $V_2 = \alpha^*$, *where* $\alpha^* > 0$ *is determined by the equation* $R'(\alpha^*) = 0$, *or to award two equal prizes,* $V_1 = V_2 = \frac{1}{2}$.

Proof. See Appendix. ■

The basic intuition behind Proposition 4.3 generalizes to the case where the designer is not constrained to award two prizes. If the cost function is "convex enough", the optimal prize structure may involve up to $k - 1$ different prizes.

Example 4.4. *Let* $k = 3$, $m = \frac{1}{2}$ *and* $F(a) = 2a - 1$ *(i.e., uniform distribution on the interval* $[\frac{1}{2}, 1]$*). Let the cost function be* $c\gamma(x) = cx^2$. *We have* $\gamma^{-1}(x) = g(x) = x^{\frac{1}{2}}$ *and* $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. *By the results in Example 3.3, we obtain:*

$$\begin{aligned}
A(c) &= -8 + 8c - 8\ln c \\
B(c) &= 16 - 16c + 12\ln c \\
B(c) - A(c) &= 24 - 24c + 20\ln c \\
g'(A(c)) &= \frac{1}{2}(-8 + 8c - 8\ln c)^{-\frac{1}{2}} \\
\int_{\frac{1}{2}}^1 (B(c) - A(c))g'(A(c))F'(c) &= \sqrt{2}\int_{\frac{1}{2}}^1 \frac{6 - 6c + 5\ln c}{\sqrt{(-1 + c - \ln c)}}dc = 0.19
\end{aligned}$$

[31] Note that the condition involves only primitives of the model: the ability distribution function, the cost function, and the number of contestants.

$B(c) - A(c)$ - thick line; $g^{'}(A(c))$ - thin line

Note how the beneficial effect of the second price (in the interval where the function $B(c) - A(c)$ is positive) gets amplified by the high values of the function g'. Numerical calculations reveal that $\alpha^* = \frac{1}{e}$ is an (approximate) solution to the equation $R'(\alpha) = 0$. Hence the optimal prize structure is $V_1 = 1 - \frac{1}{e} \approx 0.63$, and $V_2 = \frac{1}{e} \approx 0.37$. The ratio of prizes is $\frac{V_1}{V_2} = e - 1 \approx 1.71$, and the difference $V_1 - V_2$ is about one quarter of the prize sum. ■

5. Entry Fees

Contest organizers often use simple instruments such as entry fees or minimum bid requirements[32] in order to exclude low ability agents from the contest[33]. Assume then that an entry fee $E > 0$ is imposed.

[32] For example, the FCC-organized contest to set the standard for high-definition television was open to anyone with a \$200.000 entry fee (see Taylor, 1995). All professional sport competitions are restricted to athletes or teams that fulfill a certain prespecified standard.

[33] In models with endogenous participation such instruments can also control for the number of contestants. Having less than the free-entry number of participants has been shown to be optimal in the research contests studied by Taylor (1995) and Fullerton and McAfee (1999). Excluding specific participants can also be beneficial in complete information models with heterogenous agents - see Baye et.al. (1993). In a winner-take-all, all-pay auction with incomplete information, with ex-ante symmetric players and with a continuum of types it is never optimal to restrict the

Let b denote the equilibrium bid function with linear costs and no entry fee (see Proposition 3.1), and recall that b was determined using the condition that the type with lowest ability (i.e., $c = 1$) bids zero (and makes zero profit). Clearly such a type will not participate in a contest where an entry fee $E > 0$ has to be paid, and therefore we need to modify here the boundary condition.

Assume then that an entry fee $E > 0$ is imposed. Solving a differential equation which is otherwise analogous to the one in the proof of Proposition 4.1, we obtain the equilibrium of a two-prize contest where contestants have cost function $c\gamma$ and where the designer imposes an entry fee E : types in an interval $[m, c_E]$ participate[34] and bid according to the bid function $b_E(c) = \gamma^{-1}(b(c) - d)$, where $c_E \in [m, 1]$ and $d \geq 0$ are determined by the zero-bid and zero profit conditions:

$$b_E(c_E) = 0 \tag{5.1}$$

$$V_1(1 - F(c_E))^{k-1} + (k-1)V_2F(c_E)(1 - F(c_E))^{k-2} - c_E b_E(c_E) - E = 0 \tag{5.2}$$

Since by construction $b_E(c_E) = 0$ and since $b_E(c_E) = 0 \Leftrightarrow \gamma^{-1}(b(c_E) - d) = 0 \Leftrightarrow b(c_E) - d = 0$,we can re-write the above conditions as:

$$b(c_E) - d = 0 \tag{5.3}$$

$$V_1(1 - F(c_E))^{k-1} + (k-1)V_2F(c_E)(1 - F(c_E))^{k-2} - E = 0 \tag{5.4}$$

number of contestants - this follows by the analysis in Bulow and Klemperer (1996) and by the revenue equivalence theorem.

[34] We consider here the (non-trivial) case where the entry fee is not too large, so that at least some types find it optimal to participate.

The two equations above form a system of two equations in two unkowns, which can be solved to obtain the equilibrium values for c_E and d.

For linear cost functions, we have shown that a contest designer who cannot use entry fees optimizes by allocating the entire prize sum to a single first prize. An analysis which is similar to the one performed in the seminal studies of Myerson (1981) or Riley and Samuelson (1981) shows that, with linear cost functions, a contest with a single first prize and an (optimally set) entry fee[35] is revenue-maximizing among all feasible mechanisms[36].

An interesting tension arises between the award of multiple prizes and entry fees (or minimum bid requirements) since the beneficial effect of additional prizes, i.e., increased bidding by low and middle ability types, is reduced if such types are excluded. Hence, one may guess that even with convex cost functions, a second prize becomes superfluous for revenue purposes if entry fees (or minimum bids) can be imposed. We conclude this section by displaying an Example showing that this is not the case.

Example 5.1. *Let* $k = 3$, $m = \frac{1}{2}$, $F(a) = 2a - 1$ *and* $\gamma(x) = x^2$. *Let* $E \leq 1$. *By Examples 3.3,4.4 and the above remarks we obtain that, in equilibrium , types in the interval* $[\frac{1}{2}, 1 - \frac{\sqrt{E}}{2}]$ *pay the entry fee and bid according to* $b_E(c) = \sqrt{-8 + 8c - 8 \ln c - d}$, *where* $d = 8 \ln 2 - 4\sqrt{E} - 8 \ln(2 - \sqrt{E})$. *A numerical analysis reveals that the designer's payoff decreases as a function of* E. *Hence, if the designer awards a single prize the optimal entry fee is zero. But, by Example 4.4, the designer can do better than that by awarding two prizes.* ■

[35] This fee depends on the designer's valuation for the prize, on the number of contestants and on the distribution of abilities.

[36] i.e., mechanisms that are incentive compatible and individually rational.

6. Concluding Comments

We have studied the optimal prize structure in multi-prize contests where risk-neutral players have private information about their abilities. In order to maximize the expected sum of bids, the designer should organize a winner-take-all contest if contestants have linear or concave cost functions. If the contestants have convex cost functions, then two or more prizes may be optimal[37]. The right proportion between the prizes' values depends then on the number of contestants, the distribution of abilities in the population, and on the exact form of the cost function.

A relaxation of the risk-neutrality condition introduces a lot of complexity in explicit computations of equilibria. But, assuming that equilibria can be computed, we conjecture that arguments similar to those exhibited here can be used to show the optimality of several prizes for sufficiently risk-averse contestants.

Another interesting extension would be the study of several parallel contests (with potentially different prize-structures), such that agents can choose where to compete.

We come back now to Francis Galton who concluded his article with the following remark:

> "I now commend the subject to mathematicians in the belief that those who are capable, which I am not, of treating it more thoroughly, may find that further investigations will repay trouble in unexpected directions[38]" (Galton, 1902)

[37]Bulow and Klemperer (1999) report that Avinash Dixit offers a $20 prize to the student who continues clapping the longest at the end of his game theory course. In experimental tests we established that pain is a highly convex function of clapping duration, and therefore we recommend the award of several prizes.

[38]The challenge was immediately picked by the famous statistician Karl Pearson, at that time

7. References

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8. Appendix

Proof of Proposition 3.1: Assume that all contestants in the set $K\backslash\{i\}$ bid according to the bid function b, and assume that the bid function is strictly monotonic and differentiable. Player i's maximization problem reads:

$$max_x[V_1(1-F(b^{-1}(x)))^{k-1}+(k-1)V_2F(b^{-1}(x))(1-F(b^{-1}(x)))^{k-2}-cx]$$

Let y denote the inverse of b. Using strict monotonicity and symmetry, the first order condition is:

$$\begin{aligned} 1 &= -(k-1)(V_1-V_2)y'\frac{1}{y}(1-F(y))^{k-2}F'(y)- \\ & (k-1)(k-2)V_2y'\frac{1}{y}F(y)(1-F(y))^{k-3}F'(y) \end{aligned}$$

Note that the right hand side of the FOC is a function of y only[39].

[39] i.e., this is a differential equation with separated variables.

A contestant with the lowest possible ability $c = 1$ can either never win a prize (if $k > 2$) or wins for sure the second prize (if $k = 2$). Hence the optimal bid of this type is always zero, and this yields the boundary condition $y(0) = 1$.

Denote

$$\begin{aligned} G(y) &= V_1((k-1))\int_y^1 \frac{1}{t}(1-F(t))^{k-2}F'(t)dt + \\ & V_2(k-1)\int_y^1 \frac{1}{t}(1-F(t))^{k-3}[1-(k-1)F(t)]F'(t)dt \end{aligned} \tag{8.1}$$

The solution to the differential equation with the boundary condition is given by:

$$\int_x^0 dt = -G(y) \tag{8.2}$$

We obtain that $x = G(y) = G(b^{-1}(x))$, and therefore that $b = G$. Thus, the bid function of every player is given by $b(c) = A(c)V_1 + B(c)V_2$, where:

$$\begin{aligned} A(c) &= (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-2}F'(a)da \\ B(c) &= (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-3}[(k-1)F(a)-1]F'(a)da \end{aligned}$$

We now check that the candidate equilibrium function b is strictly monotonic decreasing (it is clearly differentiable). Note first that

$$A'(c) = -(k-1)\frac{1}{c}(1-F(c))^{k-2}F'(c) < 0$$

for all $c \in [m,1)$.We have also

$$B'(c) = (k-1)\frac{1}{c}(1-F(c))^{k-3}F'(c)[(1-(k-1)F(c)]F'(c)$$

Because $V_1 \geq V_2$ we obtain for all $c \in [m,1)$:

$$\begin{aligned} b'(c) &= A^{'}(c)V_1 + B'(c)V_2 \\ &\leq V_2(A'(c) + B^{'}(c)) \\ &= -V_2(k-1)(k-2)\frac{1}{c}F(c)(1-F(c))^{k-3}F'(c)) \\ &< 0 \end{aligned}$$

Assuming that all contestants other than i bid according to b, we finally need to show that, for any type c of player i, the bid $b(c)$ maximizes the expected utility of that type. The necessary first-order condition is clearly satisfied (since this is how we "guessed" $b(c)$ to start with). We now show that a sufficient second-order condition (called "pseudo-concavity") is satisfied. Let

$\pi(x,c) = V_1(1-F(b^{-1}(x)))^{k-1} + (k-1)V_2F(b^{-1}(x))(1-F(b^{-1}(x)))^{k-2} - cx$ be the expected utility of player i with type c that makes a bid x. We will show that the derivative $\pi_x(c,x)$ is nonnegative if x is smaller than $b(c)$ and non-positive if x is larger than $b(c)$. As $\pi(x,c)$ is continuous in x, this implies that $\pi(x,c)$ is maximized at $x = b(c)$. Note that

$\pi_x(x,c) = -(k-1)(V_1 - V_2)\frac{db^{-1}(x)}{dx}(1-F(b^{-1}(x)))^{k-2}F'(b^{-1}(x)) -$
$(k-1)(k-2)V_2\frac{db^{-1}(x)}{dx}F(b^{-1}(x))(1-F(b^{-1}(x)))^{k-3}F'(b^{-1}(x)) - c.$

Let $x < b(c)$, and let $\widehat{c}$ be the type who is supposed to bid x, that is $b(\widehat{c}) = x$. Note that $\widehat{c} > c$ since b is strictly decreasing. Differentiating $\pi_x(x,c)$ with respect to c yields $\pi_{xc}(x,c) = -1 < 0$. That is, the function $\pi_x(x,\cdot)$ is decreasing in c. Since $\widehat{c} > c$,we obtain $\pi_x(x,c) \geq \pi_x(x,\widehat{c})$

Since $x = b(\widehat{c})$ we obtain by the first order condition that $\pi_x(x,\widehat{c}) = 0$, and therefore that $\pi_x(x,c) \geq 0$ for every $x < b(c)$. A similar argument shows that $\pi_x(x,c) \leq 0$ for every $x > b(c)$. ■

The symmetric equilibrium with p prizes:

Fix agent i, and let $F_s(a), 1 \leq s \leq p$, denote the probability that agent i with type a meets $k-1$ competitors such that $s-1$ of them have lower types, and $k-s$ have higher types. Hence F_s is exactly the probability of winning the s'th prize[40]. We have then

$$F_s(a) = \frac{(k-1)!}{(s-1)!(k-s)!}(1-F(a))^{k-s}(F(a))^{s-1}$$

The corresponding derivatives are given by

$$F_1'(a) = -(k-1)(1-F(a))^{k-2}F'(a) \tag{8.3}$$

and by

$$\begin{aligned} F_s'(a) &= \frac{(k-1)!}{(s-1)!(k-s)!}(1-F(a))^{k-s-1}(F(a))^{s-2}F'(a) \\ &\cdot[(1-k)F(a)+(s-1)] \end{aligned} \tag{8.4}$$

for $s > 1$. Note that $A(c) = \int_c^1 -\frac{1}{a}F_1'(a)da$ and that $B(c) = \int_c^1 -\frac{1}{a}F_2'(a)da$. Analogously to the case of two prizes, the equilibrium bid for any number of prizes $p > 2$ and $k \geq p$ contestants with linear cost functions is given by:

$$b(c) = \sum_{s=1}^{p} V_s \cdot \int_c^1 -\frac{1}{a}F_s'(a)da \tag{8.5}$$

■

The following technical Lemma is important since it is repeatedly used in the proofs of Propositions 3.2, 4.2, 4.3:

[40]Recall that in equilibrium we expect i to bid more than competitors with higher types (lower ability). For the relation between the probabilities F_s and the distribution of differences of successive order statistics (which were in fact Galton's theme) see Chapter 2 in Arnold and Balakrishnan (1989).

Lemma 8.1. *Assume that there are two prizes,* $V_1 \geq V_2 \geq 0$, *and let* $b(c) = A(c)V_1 + B(c)V_2$ *be the symmetric equilibrium bid function for* $k \geq 3$ *contestants having linear cost functions. Then the following properties hold:*

1. $A(1) = B(1) = 0$

2. $\forall c \in [m, 1)$, $A(c) > 0$, and $A'(c) < 0$

3. Let c^* be such that $F(c^*) = \frac{1}{k-1}$. Then $B'(c^*) = 0$, $B'(c) > 0$ for all $c \in [m, c^*)$, and $B'(c) < 0$ for all $c \in (c^*, 1]$

4. $\mid B'(c) \mid > \mid A'(c) \mid$ for c in a neighborhood of 1.

5. $B(m) < 0$

6. For any $k > 2$,there exists a unique point $c^{**} \neq 1$ such that $A(c^{**}) = B(c^{**})$.

7. $\int_m^1 (B(c) - A(c))F'(c)dc < 0$[41]

Proof.

1. This is obvious by definition.

2. $A(c) > 0$ for $c \in [m, 1)$ is obvious by definition. Further we have $A'(c) = -(k-1)\frac{1}{c}(1 - F(c))^{k-2}F'(c) < 0$ for all $c \in [m, 1)$ and $A'(1) = 0$.

3. $B'(c) = (k-1)\frac{1}{c}(1 - F(c))^{k-3}F'(c)[(1 - (k-1)F(c)]$

   For c^* such that $F(c^*) = \frac{1}{k-1}$ we obtain $B'(c^*) = 0$. Moreover, $B'(c) > 0$ for all $c \in [m, c^*)$, and $B'(c) < 0$ for all $c \in (c^*, 1)$. Finally, $B'(1) = 0$.

[41] In order to prove that one prize is optimal even if the designer can award more than two prizes, it is enough to show that for all s, $2 \leq s \leq p$, it holds that $\int_m^1 (\int_c^1 -\frac{1}{a}(F_s'(a) - F_1'(a))da)dc < 0$ (see formulae 8.3-8.4 above). The proof here treats the case $s = 2$. The proofs for the cases $s > 2$ are completely analogous.

4. For for all $c \in [c^{*}, 1)$ we obtain that

$$\begin{aligned} & \mid B'(c) \mid - \mid A'(c) \mid \\ = & -B'(c) + A'(c) \\ = & (k-1)\frac{1}{c}(1-F(c))^{k-3}F'(c)(kF(c)-2) \end{aligned}$$

For $k > 2$ we obtain that $\mid B'(c) \mid - \mid A'(c) \mid$ is positive for c close enough to 1 (since $kF(c) > 2$ for such types.)

5. We have

$$\begin{aligned} B(m) &= (k-1)(\int_{m}^{c^{*}} \frac{1}{a}(1-F(a))^{k-3}[(k-1)F(a)-1]F^{'}(a)da \\ &\quad +(k-1)(\int_{c^{*}}^{1} \frac{1}{a}(1-F(a))^{k-3}[(k-1)F(a)-1]F^{'}(a)da \\ &< (k-1)\int_{m}^{1}(1-F(a))^{k-3}[(k-1)F(a)-1]F^{'}(a)da \end{aligned}$$

The last inequality follows by noting that the integrand in the first integral is negative and that the integrand in the second integral is positive. Thus, if we multiply both integrands by the increasing function $h(a) = a$ we strictly increase the value of the sum of the two integrals. In order to prove that $B(m) < 0$ it is then enough to prove that $\int_{m}^{1}(1-F(a))^{k-3}[(k-1)F(a)-1]F^{'}(a)da = 0$. By the change of variable $z = F(a)$,we obtain

$$\begin{aligned} & \int_{m}^{1}(1-F(a))^{k-3}[(k-1)F(a)-1]F^{'}(a)da \\ = & \int_{0}^{1}(1-z)^{k-3}[(k-1)z-1]dz = 0 \end{aligned}$$

6. This follows by combining all properties above.

7. We know that $B(c) - A(c) > 0$ for all $c \in [m, c^{**})$ and that $B(c) - A(c) < 0$ for all $c \in (c^{**}, 1)$. This yields:

$$
\begin{aligned}
& \int_m^1 (B(c) - A(c))F'(c)dc \\
= & \int_m^{c^{**}} (B(c) - A(c))F'(c)dc + \int_{c^{**}}^1 (B(c) - A(c))F'(c)dc \\
= & (k-1)\int_m^{c^{**}} [\int_c^1 \frac{(1-F(a))^{k-3}}{a}(kF(a)-2)F'(a)da]F'(c)dc + \\
& (k-1)\int_{c^{**}}^1 [\int_c^1 \frac{(1-F(a))^{k-3}}{a}(kF(a)-2)F'(a)da]F'(c)dc \\
< & (k-1)\frac{1}{c^{**}}\int_m^{c^{**}} [\int_c^1 (1-F(a))^{k-3}(kF(a)-2)F'(a)da]F'(c)dc + \\
& (k-1)\frac{1}{c^{**}}\int_{c^{**}}^1 [\int_c^1 (1-F(a))^{k-3}(kF(a)-2)F'(a)da]F'(c)dc \\
= & (k-1)\frac{1}{c^{**}}\int_m^1 [\int_c^1 (1-F(a))^{k-3}(kF(a)-2)F'(a)da]F'(c)dc \\
= & \frac{1}{c^{**}}\int_0^1 [\int_v^1 (k-1)(1-z)^{k-3}(kz-2)dz]dv = 0
\end{aligned}
$$

The last equality follows by the changes of variables $F(a) = z$ and $F(c) = v$. ∎

Proof of Proposition 4.1: As in the proof of Proposition 3.1, player i's maximization problem reads:

$$max_x[V_1(1 - F(b^{-1}(x)))^{k-1} + (k-1)V_2F(b^{-1}(x))(1 - F(b^{-1}(x)))^{k-2} - c\gamma(x)]$$

Letting y denote the inverse of b, the first order condition is:

$$
\begin{aligned}
\gamma'(x) &= -(k-1)(V_1 - V_2)y'\frac{1}{y}(1 - F(y))^{k-2}F'(y) - \\
& \quad (k-1)(k-2)V_2y'\frac{1}{y}F(y)(1 - F(y))^{k-3}F'(y)
\end{aligned}
$$

Integration and the use of the boundary condition $y(1) = 0$ yield $\gamma(x) = G(y)$, where $G(y)$ is defined exactly as in the proof of Proposition 3.1 (see equation 8.1) Hence, $x = \gamma^{-1}(G(y)) = g(G(b^{-1}(x)))$ and $b = g(G)$. The candidate equilibrium bid function $b(c) = g(A(c)V_1 + B(c)V_2)$ is strictly decreasing since, for all $c \in [m, 1)$, it holds:

$$
\begin{aligned}
&\frac{dg}{dc}(A(c)V_1 + B(c)V_2) \\
= \; & g'(A(c)V_1 + B(c)V_2) \cdot (A'(c)V_1 + B'(c)V_2) < 0
\end{aligned}
$$

The last inequality follows because $g' > 0$ by assumption, while $A'(c)V_1 + B'(c)V_2 < 0$ since this is the derivative of the bid function with linear cost functions (see proof of Proposition 3.1). For the sufficient second-order condition we proceed exactly as in the proof of Proposition 3.1.

The equilibrium bid for $p > 2$ prizes , and $k \geq p$ contestants with cost functions of the form $c\gamma(x)$ is given by $b(c) = \gamma^{-1}(\sum_{s=1}^{p} V_s \cdot \int_c^1 -\frac{1}{a} F_s^{'}(a)da)$, where the $F_s^{'}(a)$ are given in formulas 8.3 and 8.4.

■

Proofs of Propositions 4.2 and 4.3: Recall that the designer's revenue as a function of the value of the second prize is:

$$R(\alpha) = k \int_m^1 g[A(c) + \alpha(B(c) - A(c))]F'(c)dc \tag{8.6}$$

Note that

$$R'(\alpha) = k \int_m^1 (B(c) - A(c))g'[A(c) + \alpha(B(c) - A(c))]F'(c)dc \tag{8.7}$$

and that:

$$R''(\alpha) = k \int_m^1 (B(c) - A(c))^2 g''[A(c) + \alpha(B(c) - A(c))]F'(c)dc \tag{8.8}$$

Observe also that

$$
\begin{aligned}
\frac{dg'[A(c) + \alpha(B(c) - A(c))]}{dc} &= g''[A(c) + \alpha(B(c) - A(c))] \\
&\quad \cdot [(1 - \alpha)A'(c) + \alpha B'(c)]
\end{aligned} \tag{8.9}
$$

and that

$$(1-\alpha)A'(c)+\alpha B'(c)<0 \tag{8.10}$$

since this last term is the derivative of the equilibrium bid function for contestants having linear cost functions.

Concave cost functions: The cost function of contestant i with ability c, $c\gamma$, has the additional feature that $\gamma'' \leq 0$. Hence $g'' = (\gamma^{-1})'' \geq 0$. By equations 8.9 and 8.10 we obtain that the positive function $g'[A(c)+\alpha(B(c)-A(c))]$ is decreasing in c. By Lemma 8.1, there exists a unique point $c^{**} \neq 1$ such that $B(c) < A(c)$ for all $c \in [m, c^{**})$ and $B(c) > A(c)$ for all $c \in (c^{**}, 1]$. This means that in the integral defining $R'(\alpha)$, all negative terms of the form $B(c)-A(c)$ are multiplied by relatively high values of g' , while all positive terms $B(c)-A(c)$ are multiplied by relatively lower values. By Lemma 8.1-7, we obtain:

$$R'(\alpha)=k\int_m^1 g'[A(c)+\alpha(B(c)-A(c))](B(c)-A(c))F'(c)dc<0 \tag{8.11}$$

Hence, the designer's payoff function has a maximum at $\alpha = 0$, and a single prize is optimal.

Convex cost functions: The cost function $c\gamma$ has the additional feature that $\gamma'' \geq 0$. Hence $g'' = (\gamma^{-1})'' \leq 0$. By equations 8.9 and 8.10 we obtain that the positive function $g'[A(c)+\alpha(B(c)-A(c))]$ is increasing in c. This means that in the integral defining $R'(\alpha)$ all negative terms of the form $B(c)-A(c)$ are multiplied by relatively low values of g', while all positive terms $B(c)-A(c)$ are multiplied by higher values. Moreover, for all $\alpha \in [0, \frac{1}{2}]$ we have

$$R''(\alpha)=k\int_m^1 (B(c)-A(c))^2 g''[A(c)+\alpha(B(c)-A(c))]F'(c)dc \leq 0$$

If condition 4.1 is satisfied we have then:

$$R'(0) = k \int_m^1 ((B(c) - A(c))g'(A(c))F'(c)dc > 0 \tag{8.12}$$

Hence, the revenue function $R(\alpha)$ cannot have a maximum at $\alpha = 0$. It either has a maximum at α^* such that $R^{'}(\alpha^*) = 0$ or at $\alpha = \frac{1}{2}$.

For the converse, assume that two prizes are optimal. This means that $\alpha = 0$ is not a maximum of $R(\alpha)$. If condition 4.1 is not satisfied we obtain $R'(0) \leq 0$. Together with $R''(\alpha) \leq 0$ for all $\alpha \in [0, \frac{1}{2}]$ we obtain a contradiction. ■