Carrots and Sticks: Prizes and Punishments in Contests

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Abstract

We study optimal contest design in situations where the designer can reward high performance agents with positive prizes and punish low performance agents with negative prizes. We link the optimal prize structure to the curvature of distribution of abilities in the population. In particular, we identify conditions under which, even if punishment is costly, punishing the bottom is more effective than rewarding the top in eliciting effort input . If punishment is costless, we study the optimal number of punishments in the contest.

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1 Introduction

Contests are widely used to model political lobbying, sports competitions, job promotions, and R&D races. In most cases, the contest's organizer has some control over the structure of rewards and punishments: teachers determine who passes and who fails the exam, a company sets rules about promotions and layoffs, and sport governing bodies choose the number of teams advancing to a higher league and the number of those relegated to a lower one.

In this paper we characterize the optimal prize structure in a contest where the designer can use both rewards (carrots) and punishments (sticks) in order to motivate participants to exert effort.[1] This relaxes an important and restrictive assumption made in the existing literature who has only examined positive prizes. Once we allow the designer to punish players who perform poorly, an interesting question arises: when is it more effective to punish the "bottom" and when is it more effective to reward the "top"?

Punishments can be costly or costless to the contest designer, and we study here both cases. For example, in a labor tournament, the firm usually incurs costs if it wants to fire workers who perform poorly (e.g., worker compensation, production disruption, or loss of clients). It could also face replacement and search costs (e.g., a faculty denying tenure to some candidates). In other instances, punishments can be almost costless to implement (e.g., relegation in sport tournaments).

We assume that the designer has a limited budget, which means that costly punishments cannot be arbitrarily high. Subject to her budget constraint, the designer determines the number and size of prizes in order to maximize the participants' expected total effort. In particular, some prizes can be positive (rewards) or negative (punishments). Given the prize structure, players choose effort level to maximize their expected payoff.

We show that the relative effectiveness of punishments compared to rewards crucially depends on the curvature of the distribution of player abilities. Exploiting results about normalized spacings among order statistics developed by Barlow and Prochan (1966), we first revisit one of the main results in Moldovanu and Sela (2001): if punishment is not feasible, the optimal prize structure offers a single prize to the top performer. If rewards are not feasible, we then show that the designer should only punish the player with the lowest performance if the distribution of abilities has an increasing hazard (or failure) rate. If this last condition is not satisfied, more punishments may be optimal.

[1] See Arvey and Ivancevich (1980) for a comprehensive discussion about the use of punishments in organizations.

If the designer can punish as well as reward, the optimal prize structure will depend on the relative marginal cost of punishments, and on the distribution of the players' abilities. For example, if the marginal cost of punishments is the same as the marginal cost of rewards, it is optimal to punish the worst performing player if the distribution of abilities is convex. If that player is punished, even high types will work hard to avoid performing worst because they know that there is a relatively high probability that competitors also have high types.

Costless punishments are, of course, more attractive to the designer. We first look at the case where players have no exit option, and we show that the optimal number of punishments is decreasing in the convexity of the ability distribution. Intuitively, when the distribution is more convex, there are relatively more high ability players. In order to motivate participants to work hard, the designer should differentiate between players with moderate performances and low performances, rather than punishing all of them.

In many situations players have outside options, and thus they may choose not to participate in the contest. Thus, the designer needs to satisfy an individual rationality constraint (chosen here to be interim: the expected payoff of a player, conditional on his ability, must be nonnegative). We show that punishments continue to be effective even in this case since they allow the designer to exclude low ability participants, akin to entry fees or reserve prices.

The two papers most closely related to the present research are Moldovanu and Sela (2001) and Moldovanu, Sela and Shi (2007). The first develops an all-pay auction framework with incomplete information in order to study the optimal allocation of several non-negative prizes (rewards) in contests where the designer has a fixed budget[2]. Moldovanu, Sela and Shi (2007) apply the all-pay auction framework to analyze the incentive effects of social status in organization design[3]. In their model, contestants who perform poorly have a low social status, and it is shown that the designer can take advantage of the agents' status concern in order to induce high effort. That paper did not study, however, the contest design problem with a general structure of rewards and punishments.

Our paper is also related to the large literature on tournaments, initiated by Lazear and Rosen (1981). That literature has shown how prizes based on rank-orders of performance can be effectively used to provide incentives in labor tournaments (see also Green and Stokey (1983), and Nalebuff and Stiglitz (1983)). The labor tournament model has been extended to analyze political lobbying

[2] Other papers that study the effects of multiple prizes in contests are Glazer and Hassin (1988), Barut and Kovenock (1998), Clark and Riis (1998) and Moldovanu and Sela (2006).

[3] This paper introduced the Barlow-Proschan (1966) order statistics results into the mechanism design literature.

and research contests. For example, Che and Gale (1998) use an all-pay auction with complete information to model political campaigns, and show how a cap on individual political contribution may actually increase aggregate expenditures. In Taylor (1995), Fullerton and McAfee (1999) and in Che and Gale (2003) research contests are modeled as rank-ordered tournaments and it is shown that it is beneficial to exclude some participants (see also Baye, Kovenock, and de Vries, 1993). In these papers exclusion is costless for the designer.

In a recent paper, Akerlof and Holden (2007) extend the analysis of Lazear and Rosen (1981) to the case with multiple prizes. They link the optimal prize structure to the form of utility functions, and show that the prize difference between two adjacent top players is often smaller than the prize difference between two adjacent bottom players. Their model is quite different from ours: they assume that the relationship between effort and performance is stochastic, and that agents are homogeneous in abilities. Moreover, these authors do not model punishments as negative prizes.

Finally, we want to note that the experimental literature on bonuses versus fines has mostly focused on principal-agent designs (e.g, Andreoni et al., 2003, and Fehr and Schmidt, 2007). It would be very interesting to experimentally study the effects of prizes and punishments also in contest situations where several agents compete with each other.

The rest of the paper is organized as follows. Section 2 describes the basic model framework and characterizes the equilibrium. Costly punishment is studied in Section 3. The analysis of costless punishment is contained in Section 4. There we divide the analysis into two parts, according to whether the contestants can exit the contest or not. Section 5 concludes.

2 The Model

We consider a contest with n players where each player j makes an effort e_j. For simplicity, we postulate a deterministic relation between effort and output, and assume these to be equal. Efforts are submitted simultaneously. An effort e_j causes a cost of e_j/a_j, where a_j is an ability parameter.

The ability (or *type*) of contestant j is private information to j. Abilities are drawn independently of each other from the interval $[0, 1]$ according to a distribution function F that is common knowledge. We assume that F has a continuous density $f = dF > 0$.

The designer can allocate n prizes to the n players: $V_1 \leq V_2 \leq \cdots \leq V_n$. Each V_i can be positive (a reward), zero or negative (a punishment). The contestant with the highest effort wins the first prize V_n, the contestant with the second highest effort wins the second prize V_{n-1}, and so on until all the prizes are allocated. The payoff of contestant i who has ability a_i and submits effort e_i is $V_j - e_i/a_i$ if i wins prize j.

Each contestant i chooses her effort in order to maximize her expected utility (given the other competitors' efforts and the values of the different prizes). The contest designer determines the number and size of prizes in order to maximize total expected effort $\sum_{i=1}^{n} e_i$.

We use the following notations:

1. $A_{k,n}$ denotes k-th order statistic out of n independent variables independently distributed according to F. Note that $A_{n,n}$ is the highest order statistic, and so on...

2. $F_{k,n}$ denotes the distribution of $A_{k,n}$, and $f_{k,n}$ denotes its density;

3. $E(k,n)$ denotes the expected value of $A_{k,n}$, where we set $E(0,n) = 0$. Note that $E(n,n)$ is the expectation of the maximum, or highest order statistic, and so on...

2.1 Equilibrium Derivation and Total Effort

We focus here on a symmetric equilibrium. Let $\beta(a)$ denote the "bidding" strategy for the player with type a. This function relates ability to equilibrium effort. Assuming a symmetric equilibrium in strictly increasing strategies and applying the revelation principle, we can formulate the player's optimization problem as follows: player j with ability a chooses to behave as an agent with ability s in order to solve the following problem:

$$\max_s \sum_{i=1}^{n} F_i^n(s) V_i - \frac{\beta(s)}{a},$$

where $F_i^n(s)$ denotes the probability that a player's type s ranks exactly i-th lowest among n random variables distributed according to F. The first term represent the expected payment from reporting s, while the second term is the effort cost. It is easy to verify that

$$F_i^n(s) = \frac{(n-1)!}{(i-1)!(n-i)!}[F(s)]^{i-1}[1-F(s)]^{n-i}, \ i = 1, 2, ..., n.$$

Define $F_{n,n-1}(s) \equiv 0$ and $F_{0,n-1}(s) \equiv 1$ for all $s \in [0,1]$. Then it is immediate that $F_i^n(s) = F_{i-1,n-1}(s) - F_{i,n-1}(s)$. Therefore, we can rewrite the agent's maximization problem as

$$\max_s \sum_{i=1}^{n} \left[F_{i-1,n-1}(s) - F_{i,n-1}(s)\right] V_i - \frac{\beta(s)}{a}.$$

In equilibrium, the above maximization problem must be solved by $s = a$, and the solution of the resulting differential equation with boundary condition $\beta(0) = 0$ is given by

$$\beta(a) = \int_0^a x \left\{ -f_{1,n-1}(x)V_1 + \sum_{i=2}^{n-1} \left[f_{i-1,n-1}(x) - f_{i,n-1}(x) \right] V_i + f_{n-1,n-1}(x)V_n \right\} dx. \qquad (1)$$

The expected total effort is given by

$$E_{total} = n \int_0^1 \beta(a) f(a) da.$$

Note that

$$\begin{aligned} & n \int_0^1 \left[\int_0^a x f_{r,n-1}(x) dx \right] f(a) da \\ = \; & n \left[F(a) \int_0^a x f_{r,n-1}(x) dx \right]_0^1 - n \int_0^1 F(a) a f_{r,n-1}(a) da \\ = \; & n \int_0^1 a \left(1 - F(a)\right) f_{r,n-1}(a) da, \end{aligned}$$

where the first equality follows from integration by part. We further observe that

$$n \left(1 - F\left(a\right)\right) f_{r,n-1}(a) = \left(n - r\right) f_{r,n}(a).$$

Therefore, we have

$$n \int_0^1 \left[\int_0^a x f_{r,n-1}(x) dx \right] f(a) da = \left(n - r\right) E\left(r, n\right).$$

By repeatedly applying above equation, the expected total effort becomes

$$E_{total} = E_{total}(V_1, V_2, ..., V_n) = \sum_{i=1}^{n} \left[\left(n - i + 1\right) E\left(i - 1, n\right) - \left(n - i\right) E\left(i, n\right) \right] V_i. \qquad (2)$$

It is clear from equation (2) that the expected total effort (E_{total}) is linear in prizes $V_i \left(i = 1, .., n\right)$. Thus, the optimal prize structure will depend on the magnitude of the different marginal effects of each prize.

3 The Optimal Prize Structure with Costly Punishments

We begin by studying the optimal prize structure when punishment is costly to the designer, and when the designer faces a fixed budget. Throughout this section, players have no option of quitting the contest.

The designer has a budget $P < \infty$. The marginal cost of providing a unit of positive prize is normalized to 1, while the marginal cost of providing a unit of negative prize is denoted by $\gamma > 0$.

Without loss of generality, suppose there are J $(0 \le J \le n)$ negative prizes. That is, $V_j < 0$ for $j \le J$ and $V_j \ge 0$ for $j > J$. Then the designer's problem can be written as

$$\max_{\{V_1,...,V_n\}} E_{total}(V_1, V_2, ..., V_n)$$
$$s.t. \quad : \quad \gamma \sum_{j=1}^{J} (-V_j) + \sum_{j=J+1}^{n} V_j \le P.$$

The designer chooses the prize structure to maximize total expected effort subject to the constraint that the monetary cost of rewards and punishments cannot exceed the budget P. Since both the objective function and the constraint are linear in prizes, the key objects are the marginal gains in expected total effort from increasing punishments or rewards.

We start with a new, simple proof of one of the main results in Moldovanu and Sela (2001): a winner-takes-all contest is optimal among all those that award only non-negative prizes (rewards) if the effort cost function is linear in effort input

Proposition 1 *If only rewards can be allocated, then, for every distribution function F, the optimal prize structure is $V_n = P$ and $V_i = 0$ for all $i < n$.*

Proof. The marginal gain in effort by spending one more dollar in rewarding players is

$$\begin{aligned} \frac{\partial E_{total}}{\partial V_i} &= (n-i+1)\, E\,(i-1,n) - (n-i)\, E\,(i,n) \\ &= E(i,n) - \ (n-i+1)\,[E\,(i,n) - E\,(i-1,n)],\ 1 \le i \le n \end{aligned}$$

Observe that

$$\frac{\partial E_{total}}{\partial V_n} = E\,(n-1,n) \ge 0$$

and that, for $1 \le i < n$,

$$\frac{\partial E_{total}}{\partial V_n} - \frac{\partial E_{total}}{\partial V_i} = E\,(n-1,n) - E(i,n) + \ (n-i+1)\,[E\,(i,n) - E\,(i-1,n)] \ge 0$$

Thus, the designer optimally rewards only the player with the highest effort. ■

Next, we allow the designer to punish players with poor performance. It turns out that the optimal prize structure is then related to the curvature of the ability distribution F. A key concept for our analysis is the failure rate (or hazard rate) defined as follows:

Definition 1 *The failure rate (or hazard rate) of a distribution F is given by:*

$$\lambda\,(x) = \frac{f\,(x)}{1 - F\,(x)}$$

A distribution function F has an increasing failure rate (IFR) if its failure rate, $\lambda\,(x)$, is increasing.

The following results, due to Barlow and Proschan (1966), link properties of order statistics to properties of the distribution F. We will use these results repeatedly throughout the paper:

Lemma 1 *(a). Let F and G be two distributions such that $F(0) = G(0) = 0$, and let $G^{-1}F$ be convex on the support of F. Then $E_F(i,n)/E_G(i,n)$ is decreasing in i for a fixed n.*

(b). Assume that a distribution F with $F(0) = 0$ is convex (concave). Then $E(i,n)/i$ is decreasing (increasing) in i for a fixed n.

(c). Assume that a distribution F with $F(0) = 0$ satisfies IFR. Then $(n-i+1)(A_{i,n} - A_{i-1,n})$ is stochastically decreasing in i for a fixed n.

By applying Lemma 1-(c), we can prove a counterpart to Proposition 1 for the case where only punishments are allowed:

Proposition 2 *If only punishments can be allocated, and if the distribution of abilities F satisfies IFR, then the optimal prize structure is $V_1 = -\frac{P}{\gamma}$ and $V_i = 0$ for all $i > 1$.*

Proof. Recall that

$$\begin{aligned}\frac{\partial E_{total}}{\partial V_i} &= (n-i+1)\,E(i-1,n) - (n-i)\,E(i,n) \\ &= E(i,n) - \ (n-i+1)\,[E(i,n) - E(i-1,n)],\ 1 \leq i \leq n\end{aligned}$$

If the distribution F satisfies IFR, then, by Lemma 1-(c), $(n-i+1)(A_{i,n} - A_{i-1,n})$ is stochastically decreasing in i, and thus $-\ (n-i+1)\,[E(i,n) - E(i-1,n)]$ is increasing in i. Since $E(i,n)$ always increases in i, we obtain that $\frac{\partial E_{total}}{\partial V_i}$ is increasing in i when F satisfies IFR.

If the designer cannot use rewards, only negative terms of the form $\frac{\partial E_{total}}{\partial V_i}$ are relevant for the designer's optimal decision. Note that, for any distribution of abilities,

$$\frac{\partial E_{total}}{\partial V_1} = -(n-1)E(1,n) < 0$$

Since $\frac{\partial E_{total}}{\partial V_i}$ is here increasing in i, $\frac{\partial E_{total}}{\partial V_1}$ must have the highest absolute value among all negative terms. Consequently, it is most effective to only punish the player with the lowest effort. ■

It is important to note that, in contrast to Proposition 1 that holds for any distribution, Proposition 2 holds only under the IFR requirement. In this case enough players are "threatened" by the highest punishment, which makes it quite effective. The following example illustrates that it may not be optimal to punish only the player with the lowest effort if the distribution of abilities does not satisfy IFR:

Example 1 *Let* $F(x) = \sqrt[w]{x}$ *and* $\gamma = 1$. *Then* $E(i,n) = \frac{n!(w+i-1)!}{(i-1)!(n+w)!}$. *It is easy to verify that F does not satisfy IFR. Note that it is not optimal to punish only the player with the lowest effort if*

$$\begin{aligned}\frac{\partial E_{total}}{\partial V_2} - \frac{\partial E_{total}}{\partial V_1} &= 2(n-1)E(1,n) - (n-2)E(2,n) \\ &= \frac{n!w!}{(n+w)!}(2(n-1) - (w+1)(n-2)) < 0.\end{aligned}$$

The above holds, for example, if $n = 3$ *and* $w > 3$.

We now proceed to investigate the optimal prize structure when both rewards and punishments are allowed. If the distribution F satisfies IFR, then, by the above results, the optimal prize structure hinges on the comparison between rewarding the top and punishing the bottom: we only need to compare the effectiveness of the the top reward and the bottom punishment, which depends on the comparison between the absolute values of $\frac{\partial E_{total}}{\partial V_1} = -(n-1)E(1,n)$ and of $\gamma \frac{\partial E_{total}}{\partial V_n} = \gamma E(n-1,n)$:

Proposition 3 *Suppose that the distribution of abilities* F *satisfies* IFR.

1. *If* $(n-1)E(1,n) > \gamma E(n-1,n)$, *then the optimal prize structure is* $V_1 = -\frac{P}{\gamma}$ *and* $V_i = 0$ *for all* $i > 1$.

2. *If* $(n-1)E(1,n) \leq \gamma E(n-1,n)$, *then the optimal prize structure is* $V_n = P$ *and* $V_i = 0$ *for all* $i < n$.

As a special case, let us assume for the next result that the marginal cost to reward and to punish is the same, that is $\gamma = 1$. We obtain then:

Proposition 4 *Assume that* $\gamma = 1$.

1. *If the distribution of abilities* F *is convex, the optimal prize structure is* $V_1 = -P$ *and* $V_i = 0$ *for all* $i > 1$.

2. *If the distribution of abilities* F *is concave and satisfies IFR, then the optimal prize structure is* $V_n = P$ *and* $V_i = 0$ *for all* $i < n$.

3. *If* F *is concave (not necessarily IFR) then the optimal prize structure may include up to* $\frac{n}{2}$ *punishments.*

Proof. 1) By Lemma 1-(b), if F is convex then $E(i,n)/i$ is decreasing in i. Thus, we have

$$(n-1)\,E\,(1,n) > E\,(n-1,n)\,.$$

The result follows from Proposition 3, since F convex implies that F satisfies IFR.

2) If F is concave then $E(i,n)/i$ is increasing in i (Lemma 1-(b)). In particular, $(n-1)\,E\,(1,n) < E\,(n-1,n)\,$, and the result follows from Proposition 3.

3) Lemma 1-(b) yields that

$$\frac{E(i-1,n)}{E(i,n)} < \frac{i-1}{i},\ \forall i$$

Note that

$$\frac{i-1}{i} < \frac{n-i}{n-i+1} \text{ if and only if } i < \frac{n+1}{2}.$$

This yields

$$(n-i+1)E(i-1,n) < (n-i)E(i,n) \text{ for } i < \frac{n+1}{2}.$$

Therefore,

$$\frac{\partial E_{total}}{\partial V_i} = (n-i+1)\,E\,(i-1,n) - (n-i)\,E\,(i,n) < 0,\ \text{for } i < \frac{n+1}{2}.$$

■

The last result implies that, when the marginal cost of rewards and punishments are the same, the effectiveness of punishing the player with the lowest effort equals the effectiveness of rewarding the player with the highest effort for a uniform distribution of abilities.

3.1 Fixed Punishment Values

In this subsection we assume that all punishments have a fixed value for the agents, and a fixed cost for the designer. For example, think of a firm who can choose the number of workers to fire in a recession: it is reasonable to assume that the cost of each layoff is fixed for the firm (e.g., some compensation that needs to be paid to each fired worker), and for the workers themselves (e.g, the difference between a fixed wage and a lower unemployment aid).

Thus, we assume here that there are q punishments, each yielding utility $-U \leq 0$ to agents, at a cost for the designer of γU per punishment. Given our results in the previous section, we assume that only the player with the highest effort is rewarded by a prize $V_n \geq 0$. Assume also that F satisfies IFR.

Since the designer is not able to shift the entire budget to one, big punishment for the agent with the lowest performance, and since the value/cost of punishments is fixed, the interesting question here is how many punishments are optimal (if at all). The designer's optimization problem becomes then

$$\max_{\{q,V_n\}} \sum_{i=1}^{q} \left[(n-i+1)\, E(i-1,n) - (n-i)\, E(i,n) \right] (-U) + E(n-1,n)\, V_n$$
$$s.t.\ q\gamma U + V_n \leq P$$

Recall that

$$\begin{aligned} \frac{\partial E_{total}}{\partial V_i} &= (n-i+1)\, E(i-1,n) - (n-i)\, E(i,n) \\ &= E(i,n) - \ (n-i+1) \left[E(i,n) - E(i-1,n) \right],\ 1 \leq i \leq n \end{aligned}$$

is increasing in i when F satisfies IFR.

If

$$\left| \frac{\partial E_{total}}{\partial V_1} \right| \leq \gamma \left| \frac{\partial E_{total}}{\partial V_n} \right|$$

then it is optimal to have no punishments: $q = 0$, and $V_n = P$.

Otherwise , let i^* be the highest index such that

$$\frac{\partial E_{total}}{\partial V_{i^*}} \leq 0 \text{ and } \left| \frac{\partial E_{total}}{\partial V_{i^*}} \right| > \gamma \left| \frac{\partial E_{total}}{\partial V_n} \right|$$

Since $\frac{\partial E_{total}}{\partial V_i}$ is increasing in i , and since $\left| \frac{\partial E_{total}}{\partial V_1} \right| > \gamma \left| \frac{\partial E_{total}}{\partial V_n} \right|$, i^* is well defined. Note that for all $1 \leq i \leq i^*$ it must hold that $\frac{\partial E_{total}}{\partial V_i} \leq 0$. Thus, it is optimal to set i^* punishments: $q = i^*$ and $V_n = P - i^* \gamma U$.

4 The Optimal Prize Structure with Costless Punishments

We assume now that punishments are costless, that is, $\gamma = 0$. As long as the designer need not worry about the participation constraint, punishments will necessarily be part of the optimal incentive scheme.

Given our results in the previous section, we assume here that only the player with the highest effort is rewarded. We focus then on the following question: what is the optimal number of identical, costless punishments? We continue to assume here that the magnitude of the punishment is fixed and is not subject to optimization.

4.1 Contests without exit

We assume here that no player has the option to stay out of the contest. Therefore, punishing the bottom player (setting $V_1 < 0$) will always increase the total effort. Denote by $q \geq 1$ the number of punishments. By (2), total effort is given by

$$E_{total} = V_n E(n-1, n) - V_1(n-q)E(q, n). \tag{3}$$

The following proposition summarizes a general relation between the optimal number of punishments and the distribution of the players' abilities.

Proposition 5 *Assume that $\gamma = 0$, and consider two distributions of abilities G, F leading to optimal numbers of punishments q_G and q_F, respectively. If $G^{-1}F$ is convex then $q_G \geq q_F$.*

Proof. Since q_F is optimal for F, we have for all $q < q_F$,

$$(n - q_F)E_F(q_F, n) \geq (n-q)E_F(q, n) \Leftrightarrow \frac{E_F(q_F, n)}{E_F(q, n)} \geq \frac{(n-q)}{(n-q_F)}.$$

In order to show that $q_G \geq q_F$, it is sufficient to show that, under the distribution G, total effort induced by q_F punishments is larger than total effort induced by $q < q_F$ punishments. That is, we need to show that, for $q < q_F$,

$$(n - q_F)E_G(q_F, n) \geq (n-q)E_G(q, n) \Leftrightarrow \frac{E_G(q_F, n)}{E_G(q, n)} \geq \frac{(n-q)}{(n-q_F)}$$

Thus, it is sufficient to show that for $q < q_F$,

$$\frac{E_F(q, n)}{E_G(q, n)} \geq \frac{E_F(q_F, n)}{E_G(q_F, n)},$$

The last inequality follows from Lemma 1-(a). ■

A simple corollary is:

Corollary 1 *Assume that $\gamma = 0$. If the distribution of abilities F is convex (concave), then the optimal number of punishments is smaller (larger) than $n/2$.*

Proof. Let F be the uniform distribution. Total effort is given by

$$E_{total} = V_n \frac{n-1}{n+1} - V_1(n-q)\frac{q}{n+1}.$$

The optimal number of punishments q must satisfy

$$\frac{dE_{total}}{dq} = \frac{-V_1}{n+1}(n-2q) = 0.$$

Thus $q = n/2$. The result follows then by Proposition 5 . ■

When only a few players are punished, high ability contestants know that they can avoid punishment by exerting effort. Conversely, if the designer punishes most players, high ability players get discouraged: they may get punished even if they work hard. The above result follows intuitively from this observations since a convex (concave) distribution puts more (less) weight on high abilities than a uniform one.

4.2 Contests with exit

We consider now the case where only types with positive expected payoffs participate in the contest, while the other players stay out. We assume again that there is one reward $V_n \geq 0$, and $q \geq 1$ punishments, each equal to $V_1 \geq -V_n$. If only one player participates, we assume that he receives a reduced prize worth $V_n + V_1$.

Let a^* denote the type of a player who is indifferent between attending the contest and staying out. The maximization problem of a player with type a is

$$\max_s V_n F_n^n(s) + V_1 \sum_{i=1}^{q} G_i^n(s) - \frac{\beta(s)}{a}$$

where

$$G_i^n(s) = \binom{n-1}{i-1} (F(s) - F(a^*))^{i-1} [1 - F(s) + F(a^*)]^{n-i},$$

is the probability that a player with type s ranks i-th lowest among the players who enter the contest. Let

$$G(s) = F(s) - F(a^*).$$

Then, we can rewrite $G_i^n(s)$ as

$$G_i^n(s) = \binom{n-1}{i-1} G(s)^{i-1}(1 - G(s))^{n-i}.$$

If a player with type a^* enters the contest then, by the definition of a^*, she gets for sure the negative prize V_1 and wins the positive prize V_n only if all other players have lower types. That is, the cutoff a^* is given by

$$V_n F(a^*)^{n-1} = -V_1. \tag{4}$$

The necessary first-order condition for an agent's maximization problem is

$$V_n f_{n-1,n-1}(a) - V_1 (n-1) \binom{n-2}{q-1} G(a)^{q-1}(1 - G(a))^{n-q-1} f(a) - \frac{\beta'(a)}{a} = 0$$

To simplify notation, define

$$\phi_{q,n-1}(s) = (n-1)\binom{n-2}{q-1} G(s)^{q-1}[1-G(s)]^{n-q-1}.$$

Then, the equilibrium bidding function for a bidder with type $a \geq a^*$ is

$$\beta(a) = \int_{a^*}^{a} \left[V_n s f_{n-1,n-1}(s) - V_1 s \phi_{q,n-1}(s) f(s)\right] ds \tag{5}$$

where we use the fact that $\beta(a^*) = 0$. Thus, total effort is given by

$$E_{total} = n \int_{a^*}^{1} \beta(a) f(a) da.$$

Interestingly, even though the players are allowed to quit the contest, punishments have a role to play in maximizing the expected total effort from the participating players, analogously to the beneficial role of an entry fee or minimal effort requirement.

Proposition 6 *Assume that $\gamma = 0$, and assume that agents have the option not to participate in the contest, which yields zero utility. Then, it is always optimal to punish, i.e., $V_1 < 0$.*

Proof. Since a^* is given by $V_n F(a^*) = -V_1$ we obtain

$$V_n f(a^*) \frac{\partial a^*}{\partial V_1} = -1 \Leftrightarrow f(a^*) \frac{\partial a^*}{\partial V_1} = -\frac{1}{V_n}$$

Let us normalize $V_n = 1$. Total effort becomes then

$$E_{total} = n \int_{a^*}^{1} \left\{ \int_{a^*}^{a} \left[s f_{n-1,n-1}(s) - V_1 s \phi_{q,n-1}(s) f(s)\right] ds \right\} f(a) da$$

Taking the derivative with respect to V_1 yields

$$\begin{aligned} \frac{\partial E_{total}}{\partial V_1} &= -n \left\{ \int_{a^*}^{a^*} \left[s f_{n-1,n-1}(s) - V_1 s \phi_{q,n-1}(s) f(s)\right] ds \right\} f(a^*) \frac{\partial a^*}{\partial V_1} \\ &\quad + n \int_{a^*}^{1} \frac{\partial \left\{ \int_{a^*}^{a} \left[s f_{n-1,n-1}(s) - V_1 s \phi_{q,n-1}(s) f(s)\right] ds \right\}}{\partial V_1} f(a) da \end{aligned}$$

Note that

$$\begin{aligned} &\frac{\partial \left\{ \int_{a^*}^{a} \left[s f_{n-1,n-1}(s) - V_1 s \phi_{q,n-1}(s) f(s)\right] ds \right\}}{\partial V_1} \\ &= -\left[a^*(n-1) F(a^*)^{n-2} f(a^*) - V_1 a^* \phi_{q,n-1}(a^*) f(a^*)\right] \frac{\partial a^*}{\partial V_1} - \int_{a^*}^{a} s \phi_{q,n-1}(s) f(s) ds \\ &= a^*(n-1) F(a^*)^{n-2} - V_1 a^* \phi_{q,n-1}(a^*) - \int_{a^*}^{a} s \phi_{q,n-1}(s) f(s) ds \end{aligned}$$

The last equality uses $f\left(a^{*}\right)\frac{\partial a^{*}}{\partial V_{1}}=-1$.

Therefore,

$$\frac{\partial E_{total}}{\partial V_{1}}=n\int_{a^{*}}^{1}\left\{a^{*}\left(n-1\right)F\left(a^{*}\right)^{n-2}-V_{1}a^{*}\phi_{q,n-1}\left(a^{*}\right)-\int_{a^{*}}^{a}s\phi_{q,n-1}\left(s\right)f\left(s\right)ds\right\}f(a)da$$

By condition (4), when $V_1=0$ we must have $a^{*}=0$. Thus

$$\frac{\partial E_{total}}{\partial V_{1}}|_{V_{1}=0}=-n\int_{0}^{1}\left\{\int_{0}^{a}s\phi_{q,n-1}\left(s\right)f\left(s\right)ds\right\}f(a)da<0.$$

Therefore, punishments are always efficient even in the contest with exit. ■

Example 2 *Consider a contest with two players who have the option not to participate. Obviously, at most one player can be then punished. Noting that* $\phi_{1,1}\left(s\right)=1$, *we get* $\beta(a)=\int_{a^{*}}^{a}\left(V_{n}-V_{1}\right)sf(s)ds$, *where* a^{*} *is given by* $V_{n}F(a^{*})=-V_{1}$. *Total effort is given by*

$$\begin{aligned}E_{total} &= 2\int_{a^{*}}^{1}\left[\int_{a^{*}}^{a}(V_{n}-V_{1})sf(s)ds\right]f\left(a\right)da\\ &= 2\int_{F^{-1}\left(\frac{-V_{1}}{V_{n}}\right)}^{1}\left[\int_{F^{-1}\left(\frac{-V_{1}}{V_{n}}\right)}^{c}(V_{n}-V_{1})sf(s)ds\right]f\left(a\right)da.\end{aligned}$$

Suppose now that F *is uniform on* $[0,1]$ *, and let* $V_n=1$. *Then*

$$E_{total}=2\int_{-V_{1}}^{1}(\int_{-V_{1}}^{a}(1-V_{1})sds)da=\frac{1}{3}\left(V_{1}+1\right)^{2}\left(2V_{1}^{2}-3V_{1}+1\right),$$

which is maximized for $V_1=-0.16$.

Finally, we want to note that, in some situations, other mechanisms for exclusion may perform better than punishments. For example, punishments that are not pecuniary (i.e., negative prizes that are not paid to the designer) are dominated by a contest with an entry fee that the designer can collect. This can be easily seen by setting an entry fee that excludes the same set of types as a given negative prize.

5 Concluding Remarks

We have investigated the combined effect of prizes and punishments in contests, and we derived the optimal prize structures in several environments with both costly and costless punishment. Our main results link the prize structure to features of the distribution of ability in the population, and to the marginal costs of rewards versus punishments. In particular, we give conditions under which rewarding the top performers is more (less) effective than punishing the worst performers. Finally, we have shown that punishments may have a beneficial effect on effort even if contestants have the option not to participate. Our present analysis completes the picture versus a large literature that has only considered positive prizes (rewards).

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