Competing Auctions with Endogenous Quantities

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Abstract

We study models where two sellers simultaneously decide on their discrete supply of a homogenous good. There is a finite, not necessarily large, number of buyers who have unit demand and privately known valuations. In the first model, there is a centralized market place where a uniform auction takes place. In the second model, there are two distinct auction sites, each with one seller, and buyers decide where to bid. Our results shed some light on the conditions leading to either the emergence of dominant marketplaces or to the coexistence of several competing sites. Using the theory of potential games, we show that in the one auction site model there is always an (almost symmetric) equilibrium in pure strategies. This equilibrium approximates the Cournot outcome as the number of buyers becomes large. In contrast, if the distribution of buyers values has an increasing failure rate, and if the marginal cost of production is relatively low, there is no pure strategy equilibrium where both sellers make positive profits in the competing sites model. We also identify conditions under which an equilibrium with a unique active site exists. Technically, we are able to deal with the finite and discrete models by using several results about order statistics developed by Richard Barlow and Frank Proschan (1965, 1966, 1975).

1 Introduction

We study models where two sellers simultaneously decide on their (discrete) supply of a homogenous good, and where there is a finite, not necessarily large, number of buyers who have unit demand and privately known valuations.[1] In the first model, the sellers bring their supply to a centralized market place where a uniform auction takes place. Thus, by the intrinsic rules of the mechanism, all sold units command the same price. In the second model, there are two distinct auction sites, each with one seller. Before observing their value, but after observing the respective supplied quantities at each site, buyers decide which auction to attend (again, each auction is a uniform price one). In principle, each auction may have its own equilibrium price.

Our results shed some light on the conditions leading to either the emergence of one dominant marketplace or the coexistence of several competing sites. In the one auction site model there is always an (almost symmetric) equilibrium in pure strategies. This equilibrium approximates the Cournot outcome as the number of buyers becomes large. In contrast, if the distribution of buyers values has an increasing failure rate (a condition assumed in most of the literature, which implies that marginal revenue is decreasing), and if the marginal cost of production is relatively low, there is no pure strategy equilibrium where both sellers make positive profits in the competing sites model. In other words, two distinct auction sites cannot coexist in equilibrium under these conditions. Coexistence becomes possible only if the marginal cost of production is sufficiently high. We also identify conditions under which an equilibrium with a unique active site exists.

Another goal of this paper is to revisit several classical scenarios in monopoly and oligopoly theory. These theories discard the standard assumption of competitive analysis concerning the large number of producing firms (sellers), but they keep the assumption that the number of consumers (buyers) is large. Our model has both a small number of sellers with endogenous supply, and a small number of buyers.[2] This framework is appealing in many situations, in particular in markets for inputs where both upstream and downstream markets are relatively concentrated oligopolies. Technically, we are able to deal with the finite and discrete models by using several results about order statistics developed by Richard Barlow and Frank Proschan (1965, 1966, 1975).[3]

One way to unify the two main models of the paper is in terms of buyers' switching cost. If the switching cost is high, that is, once buyers decide to place a bid in one auction they cannot switch to the other, we obtain the model with two competing auction sites. If the switching cost is low, then an ascending clock auction a la Demange-Gale-Sotomayor (this is a variant with money on Gale-Shapley's

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[1] Most qualitative results can be generalized to settings with more sellers.

[2] The parallels between a single-good auction theory and monopoly are well-known - see Bulow and Roberts (1989). Hansen (1988) studies a model where sellers compete for the right to become the sole supplier to a unique buyer whose purchase depends on the winning bid.

[3] For other applications of Barlow and Proschan's results see Moldovanu, Sela and Shi (2005) and Hoppe, Moldovanu and Sela (2006).

deferred acceptance algorithm) where buyers shift demand to the cheaper unit, leads to an efficient outcome (conditional on supplied quantities). This outcome is equivalent to the one of the uniform price auction in the single site model - in particular, all units are sold at a single market price.

Viewed in this perspective, the competing auctions model is one where, in the first (quantity-setting) stage, the sellers compete for market share, i.e., for the buyers that will become locked-in at the second stage (see Klemperer, 1987 for an early analysis of the forces at play in such models). By increasing quantity, a seller attracts more buyers.

"Lock-ins" appear in many settings and have a variety of reasons such as transaction costs associated with operating via several technologically different platforms/clearing houses, uncertainty about the quality/quantity offered at other marketplaces, "loyalty" contracts with operators, ex-ante market specific investments, and various forms of network externalities.

On the other hand, a model with a centralized auction site and with endogenous supply is better suited for the study of some modern markets where all transactions are executed at market price, i.e., order driven periodic auctions in financial markets, or markets for inputs such as electricity or gas.[4] At online auction sites such as eBay there are often several simultaneous auctions for identical commodities (e.g., CPU's). Sajid et al. (2004) empirically study such parallel auctions and find that a significant proportion of bidders are indeed active across the competing auctions: they repeatedly place bids in the auctions with the lowest standing bid. Moreover, prices tended to be uniform across auctions.

In contrast to our paper, most of the papers in the relatively thin literature on competing auctions consider models with several sellers endowed with a single unit of a homogenous good, and with several buyers with unit demand who decide which auction to attend. Thus, total demand and supply are fixed exogenously, whereas total supply is endogenous in our models. Moreover, most of these papers use some kind of "large market assumption" for their main results (i.e., individual agents ignore the effect of their own actions on prices) while such an assumption is not needed here. It is not always clear whether such assumptions are consistent with a limit obtained in small markets of increased size.

Peters and Severinov (2006) consider several sellers who set reserve prices at their own auctions sites and then conduct second price auctions. A large number of buyers with unit demand can move among auction sites. The main consequence of this freedom (i.e., absence of switching costs) is that there exists an efficient equilibrium (conditional on reserve prices). This is consistent with the insight obtained from the models that mimic features of the Gale-Shapley deferred acceptance algorithm (such as our model of competing sellers at a unique auction site) and should be contrasted with earlier models such as McAfee (1993)[5] or Peters and Severinov (1997) where buyers can place only one bid, thus

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[4] Lengwiler (1999), Back and Zender(2001) and McAdams (2006) analyze endogenous supply in monopolistic models where the seller adjusts supply *after* seeing the realization of demand, and where bidders demand several units. LiCalzi and Pavan (2006) allow the monopolist to ex-ante commit to a supply function. This is in the spirit of Klemperer and Meyer (1989) who study supply-function equilibria in oligopoly with demand uncertainty.

[5] He considers several sellers offering auctions at fixed sites (each seller at another site). His analysis assumes that the

leading to an inefficiency stemming from the coordination problem.

The role of a unique marketplace is also the theme of Moreno and Ubeda (2006) who introduce an explicit element of price formation in the traditional Cournot oligopoly story (while keeping the assumption that there is a large number of buyers). Their starting point is the classical paper by Kreps and Scheinkman (1983) who study a two stage game, with capacity choice in the first stage, and Bertrand competition in the second stage. Kreps and Scheinkman show that the unique equilibrium coincides with the Cournot outcome. Moreno and Ubeda (2006) look at the two stage model where firms first choose capacity and then set reservation prices at which they are willing to sell their entire capacity. The ensuing supply function is used to clear the market. The main difference to the Kreps and Scheinkman analysis is that all supply is sold at market price whereas, in Kreps and Scheinkman's model each firm charges its own price. This allows them to avoid several difficulties appearing in the Kreps-Scheinkman model.[6]

If we introduce reserve prices in our model of competing sellers at one auction site, and if we let the number of buyers go to infinity (which is the standard assumption in the literature) it can be shown that, if quantity is optimally adjusted to the number of buyers, the marginal gain from a reserve price goes to zero.[7] The limit outcome is then the classical Cournot one - corresponding to the main result of Moreno and Ubeda (2006).

An interesting strand that allows for both a small number of buyers and a small number of sellers is the literature on the so-called *strategic market games*, following the pioneering work by Shapley and Shubik (1975). In these models both buyers and sellers bid quantities, and transactions are cleared at a price equal to the ratio of demand to supply (thus, commodities are allocated in proportion to bids). Almost the entire literature identifies the number of trading sites (or "posts") with the number of traded commodities. In other words, there is a unique trading site for each traded commodity where all demand and supply of that commodity is cleared. A notable exception is Koutsougeras (2003) who allows for two separate trading sites for the same good. In his model traders can send quantity bids to either one, or to both sites. He shows via an example that the "Law of One Price" need not hold in equilibrium. Although different trading prices open up arbitrage opportunities for agents who can shift their demand/supply (which seems inconsistent with the equilibrium idea), these opportunities

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economy is large, and therefore it ignores the effects of individual buyers on prices. In particular, it ignores the fact that a change in one seller's mechanism affects the distribution of buyers at his own and at other sites.

[6] The K-S result critically depends on a particular rationing rule used when capacity is binding. Davidson and Deneckere (1986) show that the Cournot outcome cannot be the equilibrium of the two stage game for any other rationing rule. Moreover, for a wide range of capacity choices, the equilibrium at the price competition stage is in mixed strategies, leading to ex-post regret.

[7] This result is based on an analogous observation for a monopolistic firm that optimally adjusts quantity in response to a varying number of buyers. Since an increase (decrease) in quantity causes a decrease (increase) in price, a binding reserve price becomes superfluous as an instrument to control price. This should be contrasted to what happens if competing sellers have only one unit to sell (Burguet and Sakovics, 1999)

disappear when individual agents (who have, relatively speaking, a "large" impact if the number of traders is small) try to take advantage of them. Such phenomena, which also appear in our model, nicely illustrate the caution that needs to be employed when dealing with models with a small number of traders. In particular, market structure may matter a lot: in our model, even mere equilibrium existence is affected by it.

Our model of competing auction sites with two possibly distinct prices is related to that of Ellison, Fudenberg and Mobius (2004). These authors study a situation where both sellers and buyers (each with unit demand/supply) simultaneously choose one of two auction sites.[8] Thus, both buyers and sellers are locked-in at their respective sites, and both total supply and demand in the market are fixed. Prices are determined by uniform-price auctions at each site (and thus by the ratio of buyers to sellers at each location).[9] Their elegant analysis ignores the integer constraints, and some results hold only for markets with large numbers of traders.[10] Ellison et al. focus on the conditions leading to the existence of equilibria where two auction sites of unequal size can coexist. Their solved examples involve the uniform and exponential distributions of values — thus both display an increasing failure rate. Their coexistence results should be contrasted to ours under this assumption. The difference is mainly due to the endogenous supply feature in our model.

In a different vein, Neeman and Vulkan (2001) study the coexistence of two forms of trade: direct bilateral negotiations and centralized markets.[11] In their model, buyers with a high willingness to pay and sellers with low costs prefer to trade through a centralized market. This leads to an unraveling of direct negotiations, so that ultimately, all traders opt for trading through the centralized market. Their analysis uses a reduced form formulation and depends on the assumption that there are large numbers of traders.

The present paper is organized as follows. In Section 2 we describe the basic model ingredients and introduce useful definitions and notation. In Section 3 we study the case of one auction site. In Subsection 3.1 we start with a monopolistic seller using a uniform price auction. We focus on the optimal supplied quantity, and we show that it increases in the convexity of the distribution function of buyers' valuations. This allows us to derive optimal quantity estimates for large, non-parametric families of distributions (i.e., with monotone hazard rates, concave, convex) . In subsection 3.2 we remain in the single auction-site model, but add competition among sellers: each of two ex-ante symmetric sellers provides the auctioneer with several units of a homogenous good, and the auctioneer sells the total quantity via a uniform price auction. With a large number of buyers, this is equivalent to classical Cournot competition. Using the potential game approach (due to Monderer and Shapley, 1996) we

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[8] This is an application of a more general framework developed by Ellison and Fudenberg (2002)

[9] We could have also presented our model by assuming that: 1) every unit is owned by a different seller; 2) the operator of each auction site restricts entry of sellers in order to maximize total profit at the site.

[10] Anderson et al. (2005) retake this analysis while incorporating the integer constraints since ignoring them may lead to inconsistent results.

[11] Genesove (1995) offers an insightful study of such an arrangement in the New England fish industry.

prove the existence of pure-strategy Nash equilibria for any constant marginal cost. Moreover, we show that an "almost" symmetric equilibrium, where the supplied quantities differ by at most one unit, always exists. Finally, we use the above described estimates in the monopoly case to give estimates for the total quantity under competition. In Subsection 3.3 we offer an example illustrating that a merger of auction sites may not be profitable if sellers optimize supply before and after a merger (although a merger is always efficiency increasing). This contrasts an earlier insight due to Schwartz and Ungo (2002) who considered sellers with exogenously given quantities.

In Section 4 we turn to the model with two distinct auction sites, each with one seller: First, the two sellers simultaneously choose the number of units supplied in their respective auctions. Before they learn their valuations, but after they observe the sellers' decisions, buyers choose which auction to attend. In the last stage, each buyer learns his valuation and submits a bid in his selected auction. In Subsection 4.1 we study the possibility of coexistence of two auction sites. A main result is that if the marginal production cost is low enough, and if the distribution of buyers' values has an increasing hazard rate then there is no pure strategy equilibrium in which both sellers are active and make positive profits. The reason is that each seller continually increases supply in order to snatch buyers from the other site. A coexistence equilibrium is possible only if the marginal cost of production is high enough. Then, the unilateral incentives to increase supply are capped by the production cost. In Subsection 4.2 we study equilibria where only one auction site is active (i.e., the other seller prefers to stay out). Such an equilibrium is possible if the optimal monopolistic quantity produced by the active seller is high enough to deter entry — this happens, for example, if the distribution of buyers' values is sufficiently convex, and if the marginal cost is not too high.

In Section 5 we compare the equilibrium quantities and prices across three different models (monopoly, competition at one auction site, and two competing auction sites) under the assumption that there is a large number of buyers. We show that, under a mild condition, the monopoly model has the highest price and lowest quantity in equilibrium. Furthermore, if the equilibrium supply is relatively high (e.g., if the production cost is low and the distribution of buyers' valuations is sufficiently convex), the model with two auction sites is more competitive than the model with one auction site: the equilibrium price is lower and the equilibrium quantity is higher in the former model. Finally, if the equilibrium supply is relatively low, then the reverse is true.

Section 6 contains several concluding comments. Appendix A contains several useful results from the literature on order statistics. Appendix B contains all proofs.

2 The Model

There are two competing sellers and $n \geq 2$ buyers. The sellers can each produce several units of a homogenous good. Each buyer has unit demand. The valuation of buyer j for a unit of the good is private information to j. Valuations are drawn independently of each other from the interval $[0, T]$, $T \leq \infty$, according to a distribution function F that is common knowledge.

We consider two different competition models among the sellers:

1. One auction site: Sellers 1 and 2 simultaneously provide k_1 and k_2 units to the auctioneer, respectively. Then the auctioneer sells the $(k_1 + k_2)$ units via a uniform price auction. The revenue of each seller is the product of the equilibrium price and his supplied quantity.

2. Two competing auction sites: In the first stage, sellers 1 and 2 simultaneously choose the number of objects for sale k_i, $i = 1, 2$, in their own auctions. In the second stage, before they learn their valuations, but after they know the decision made by the sellers in the first stage, buyers choose whether to attend auction 1 or 2. In the third and last stage, each buyer learns his valuation and submits a bid in his selected auction (each of the two auctions is a uniform price auction). The revenue of each seller is the product of his supplied quantity and equilibrium price in his own auction.

2.1 Notation

We use the following notation: $X_{k,n}$ denotes k-th order statistic out of n independent valuations distributed according to F (note that $X_{n,n}$ is the highest order statistic, and so on..). The distribution $F_{k,n}$ and the density $f_{k,n}$ of $X_{k,n}$ are given by

$$F_{k,n}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j [1 - F(x)]^{n-j}$$

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} [1 - F(x)]^{n-k} f(x)$$

We denote by $EX_{k,n}$ the expected value of $X_{k,n}$.

The failure rate (or hazard rate) of a distribution F is defined by:

$$\lambda(x) = \frac{f(x)}{1 - F(x)}$$

A distribution function F has an *increasing failure rate* (*IFR*) if its failure (or hazard) rate, $\lambda(x)$, is increasing. Analogously, F has a *decreasing failure rate* (*DFR*) if $\lambda(x)$ is decreasing. Note that convexity of F implies IFR, while DFR implies concavity. The only distribution that is both concave and convex is the uniform, while the only distribution that is both IFR and DFR is the exponential.

A distribution function F is *star-shaped* on $[0, T]$ if $F(x)/x$ is increasing in x. Let $X(Z)$ have distributions $F(G)$ such that $F(0) = G(0) = 0$. Distribution F *is star-shaped with respect to G* if the function $G^{-1}F(x)$ is star-shaped. Notice that :

1. If $F(0) = 0 = G(0)$, $G^{-1}F$ convex implies $G^{-1}F$ star-shaped.

2. If G is the exponential distribution, then $G^{-1}F$ convex is equivalent to F being IFR.

3. If G is uniform, then $G^{-1}F$ convex is equivalent to F is convex.

3   One Auction Site

We discuss first the model of a uniform price multi-object auction with n buyers and a single, monopolist seller with variable supply. The derived insights will be repeatedly used in the analysis of competition among sellers below.

3.1   Monopoly seller

The seller decides on the supply k, and then the buyers decide what to bid in an uniform price auction for k units. We do not explicitly analyze here the use of exclusion tools such as reserve prices and entry fees. Thus all n buyers have an incentive to participate in the auction. Note that in an auction with endogenous quantity and large number of buyers, the marginal gain from setting optimal reserve price is small if quantity is optimally set. The intuition is provided by the fact that the realizations of $X_{n-k.n}$ are very close to $EX_{n-k,n}$ if n is large. The seller can control $EX_{n-k,n}$ by varying the quantity k. If the reserve price is binding and enhancing revenue, the seller can always set the quantity to k' such that, approximately, $EX_{n-k',n} = r$. After this adjustment, the reserve price is redundant.

We assume first that the seller has a zero cost of producing the object. If the seller offers $k > n$ objects, it is obvious that the equilibrium price is zero, and the seller has zero profit. Thus, without loss of generality, we restrict attention to the case $k \leq n$. In the auction, each of the k highest buyers wins a single unit, and pays the equilibrium price $X_{n-k,n}$. In expectation, the seller's maximization problem is given by

$$\max_k R(k, n) = \max_k k EX_{n-k,n}$$

The relation between the optimal number of objects and the distribution of the buyers' valuation is as follows:

Proposition 1 *Consider two distributions G, F leading to optimal supplies $k_G^*(n)$ and $k_F^*(n)$, respectively, when there are n buyers. If $G^{-1}F$ is star-shaped then $k_G^*(n) \leq k_F^*(n)$.*

Proposition 1 implies the following two corollaries:

Corollary 1 *If F is IFR (DFR), and n is large, then $k_F^*(n)$, the optimal supply in an auction with n buyers satisfies $k_F^*(n) \geq (\leq) \frac{n}{e}$, where e is the natural logarithm basis.*

Corollary 2 *If F is convex(concave), then $k_F^*(n)$, the optimal supply in the auction with n buyers satisfies $k_F^*(n) \geq (\leq) \frac{n}{2}$.*

The following result considers the function $k_F^*(n)$ that relates the optimal supply to the number of buyers. It shows that the "slope" of this function is less than unity if F is IFR (thus, the difference $n - k^*(n)$ weakly increases in n)

Proposition 2 *Let $k_F^*(n)$ be the optimal number of objects in the auction with n buyers. If F is IFR, then, for all n, $k_F^*(n+1) - k_F^*(n) \leq 1$.*

Remark 1 *When n is large we have the approximation*

$$EX_{n-k,n} \approx F^{-1}\left(\frac{n-k}{n}\right)$$

Hence, the seller's revenue in our auction is

$$R_1(k,n) = kEX_{n-k,n} \approx kF^{-1}\left(\frac{n-k}{n}\right) = kF^{-1}\left(1 - \frac{k}{n}\right)$$

Consider now a monopolist seller facing n buyers, each with unit demand and valuation distribution F. The monopolist chooses p to maximize his expected revenue.

$$R_2(p,n) = np\left[1 - F(p)\right]$$

Setting $k = n\left[1 - F(p)\right]$, or equivalently $p = F^{-1}\left[1 - \frac{k}{n}\right]$, we see that the monopoly and auction models are equivalent if the number of buyers is sufficiently large. In particular, p^*, the limit equilibrium price in a single auction with endogenous quantity and a sufficiently large number of buyers is given by

$$\lambda(p^*) = \frac{f(p^*)}{1 - F(p^*)} = \frac{1}{p^*}$$

Note that an increase of the distribution of the buyers' valuations in the hazard rate order (which implies first-order stochastic dominance) yields a higher equilibrium price.

Example 1 *Let $F(x) = x^{1/a}$ $(a > 0)$ with support $[0,1]$. When n is large, the seller's revenue if he sells k objects is*

$$R(k,n,a) = kEX_{n-k,n} \approx kF^{-1}\left(\frac{n-k}{n}\right) = k\left(\frac{n-k}{n}\right)^a$$

The optimal k is

$$k_F^*(n) = \frac{1}{a+1}n$$

and we have $k_F^*(n,a)^* < n/2$ if $a > 1$ and $k_F^*(n,a) > n/2$ if $a < 1$. The limit equilibrium price is $p^*(a) = \left(\frac{a}{a+1}\right)^a$.

Assume now that the seller faces a constant marginal cost of producing one unit, and denote this cost by $c \geq 0$. The seller's maximization problem is

$$\max_k \pi\left(k, n\right) = \max_k R\left(k, n\right) - ck = \max_k k(EX_{n-k,n} - c)$$

The marginal revenue from an additional object is given by $R(k+1, n) - R(k, n)$. The following result follows from the observation that the marginal revenue is decreasing in the number of objects if F is IFR.

Proposition 3 *If F is IFR, the optimal number of units $k_F^*(n, c)$ decreases in c for fixed n.*

3.2   Competition among sellers at a unique auction site

We consider now two sellers competing to sell units to n buyers through a single, uniform-price auction. Sellers 1 and 2 provide k_1 and k_2 objects to the auctioneer, respectively. Then the auctioneer sells $(k_1 + k_2)$ through the uniform price auction. Throughout of this section, we assume for simplicity that $c = 0$. All main results can be immediately extended to the case of positive marginal costs. Moreover, the main result, existence of equilibria of pure strategies generalizes to a setting with several sellers.

If (k_1^*, k_2^*) is an equilibrium, then for all $s \in \{-k_i^*, ..., n - k_i^* - k_j^*\}$ it must hold:

$$k_i^* EX_{n-k_i^*-k_j^*,n} - (k_i^* + s)EX_{n-k_i^*-k_j^*-s,n} \geq 0 \ , \ i = 1, 2 \tag{1}$$

The following example illustrates several interesting phenomena: **1)** Symmetric and asymmetric pure strategy equilibria may coexist; 2) Symmetric pure strategy equilibria may not exist;[12] **3)** As the number of buyers gets large, all equilibria converge to the symmetric one.

Example 2 *Suppose that F is uniform. Given k_2, firm 1 chooses k_1 to maximize*

$$R_1 = k_1 EX_{n-(k_1+k_2),n} = \frac{k_1\left(n - k_1 - k_2\right)}{n+1}$$

If k_1 is optimal, then for any $s = -k_1, ..., n - k_1 - k_2$

$$\begin{aligned} k_1\left(n - k_1 - k_2\right) - \left(k_1 + s\right)\left(n - k_1 - k_2 - s\right) &\geq 0 \\ s\left(s - n + 2k_1 + k_2\right) &\geq 0 \end{aligned}$$

Setting $s = 1, -1$, we get

$$\frac{1}{2}\left(n - k_2 - 1\right) \leq k_1 \leq \frac{1}{2}\left(n - k_2 + 1\right) \tag{2}$$

Similarly, if k_2 is optimal for firm 2, we obtain:

$$\frac{1}{2}\left(n - k_1 - 1\right) \leq k_2 \leq \frac{1}{2}\left(n - k_1 + 1\right) \tag{3}$$

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[12] Amir and Lambson (2000) exibit a sufficient condition for the existence of a unique equilibrium in the standard model of Cournot competition.

Combining equation (2) and (3), we get

$$k_1 \geq \frac{1}{2}(n - k_2 - 1) \geq \frac{1}{2}\left(n - \frac{1}{2}(n - k_1 + 1) - 1\right) \Rightarrow k_1 \geq \frac{1}{3}(n - 3)$$

$$k_1 \leq \frac{1}{2}(n - k_2 + 1) \leq \frac{1}{2}\left(n - \frac{1}{2}(n - k_1 - 1) + 1\right) \Rightarrow k_1 \leq \frac{1}{3}(n + 3)$$

Therefore,

$$\frac{n}{3} - 1 \leq k_1 \leq \frac{n}{3} + 1 \qquad (4)$$
$$\frac{n}{3} - 1 \leq k_2 \leq \frac{n}{3} + 1$$

1) *If $n = 3m$ there are three possible pure strategy equilibria (see details in the Appendix)*

$$k_1^* = m + 1, k_2^* = m - 1$$
$$k_1^* = m - 1, k_2^* = m + 1$$
$$k_1^* = k_2^* = m$$

2) *If $n = 3m + 1$, there are also three pure-strategy equilibria*

$$k_1^* = m + 1, \ k_2^* = m$$
$$k_1^* = m, \ k_2^* = m + 1$$
$$k_1^* = k_2^* = m$$

3) *If $n = 3m + 2$, there are only two asymmetric pure strategy equilibria*

$$k_1^* = m, \ k_2^* = m + 1$$
$$k_1^* = m + 1, \ k_2^* = m$$

We turn now to a proof of existence of pure-strategy Nash equilibria. Since the strategy spaces are discrete and since the payoff functions are not continuous, it is obvious that the standard approach cannot work. Fortunately, it turns out that our game is an *ordinal potential game* in the sense of Monderer and Shapley (1996). Thus it possesses a Nash equilibrium in pure strategies, corresponding to a maximizer of the potential function. The next two results hold also for the case where $c > 0$.

Theorem 1 *A pure strategy Nash equilibrium always exists in the competition model with one auction site.*

By taking a closer look at the potential function, we also get the existence of an "almost" symmetric, pure-strategy Nash equilibrium:

Definition 1 *An equilibrium (k_1^*, k_2^*) is quasi-symmetric if*

$$|k_1^* - k_2^*| \leq 1.$$

Theorem 2 *A pure strategy, quasi-symmetric Nash equilibrium always exists in the competition model with one auction site.*

Remark 2 *Consider now a large number of buyers. If the sellers supply k_1 and k_2 units, respectively, the revenues are approximated by*

$$\pi_i = k_i E X_{n-(k_i+k_j),n} \approx k_i F^{-1}\left(\frac{n-k_i-k_j}{n}\right) = k_i F^{-1}\left(1 - \frac{k_i+k_j}{n}\right)$$

This is equivalent to a standard Cournot model where the inverse demand function is given by

$$P(k_1+k_2) = F^{-1}\left[1 - \frac{k_1+k_2}{n}\right]$$

In particular, under the conditions of Amir and Lambson(2000), in the limit as n gets large, there exists a unique symmetric equilibrium.

We conclude this Section with the following equilibrium characterization result:

Proposition 4 *1) Let F be convex (concave). Then, in any equilibrium, the total supplied quantity is higher (lower) than $\frac{2}{3}n$. In particular, if the distribution of buyers' valuations is uniform, then the total supplied quantity is $\frac{2n}{3}$ units. 2) If F is IFR (DFR) , and if n is large, then in any equilibrium, the total supplied quantity is higher (lower) than $\frac{2}{1+e}n$. In particular, if the distribution of buyers' values is exponential, the supplied quantity is $\frac{2}{1+e}n$ units.*

3.3 Merging auction sites

Schwartz and Ungo (2002) consider a model where two monopolistic auction houses with separate sets of buyers decide to merge, and show that such a merger always increases total revenue (it is obvious that a merger increases total welfare since it yields an overall efficient allocation of the units). In their model, the supplied quantities before the merger are exogenous, and the merged entity simply offers the sum of the ex-ante quantities to the united set of buyers. In other words, the sellers do not optimize their supply, and, moreover, do not adjust it after the merger. In the following example, we look at the case of two separate auction sites, each with one seller. After a merger of the auction sites, the sellers remain separate and compete by optimizing the quantity supplied to the merged site. We show that, unless the number of buyers is small, the post-merger total revenue goes down, which implies that at least one of the sellers loses from the merger. Thus, efficiency gains are per-se, not enough to explain mergers of auction sites if we assume that multiple competing sellers use the merged site and that the site operator earns a fixed fraction of revenue. This conforms to the usual intuition from Cournot analysis, but our results also identifies the role played in this argument by a large number of buyers.

Example 3 *Assume that F is uniform on $[0,1]$ and assume that, before a merger, the two auction sites face $n_1 = n_2$ buyers. For the uniform distribution we know that the optimal supply at each site is $\frac{n_i}{2}$,*

$i = 1, 2$. *Thus total profit is given by:*

$$\pi_1 + \pi_2 = \frac{n_1^2/4}{n_1 + 1} + \frac{n_2^2/4}{n_2 + 1} = \frac{n_1^2}{2(n_1 + 1)}$$

After the merger, the unique auction site faces $2n_1$ buyers. For the uniform distribution, we know that the equilibrium total supply in the competition model equals $\frac{4n_1}{3}$, two-thirds the number of buyers. Thus, total revenue is

$$\pi_1^c + \pi_2^c = \frac{4}{3}n_1 \frac{2n_1 - \frac{4}{3}n_1}{2n_1 + 1} = \frac{8n_1^2}{9(2n_1 + 1)}$$

The revenue difference is

$$\pi_1^c + \pi_2^c - (\pi_1 + \pi_2) = \frac{8n_1^2}{9(2n_1 + 1)} - \frac{n_1^2}{2(n_1 + 1)} = n_1^2 \frac{7 - 2n_1}{18(2n_1 + 1)(n_1 + 1)}$$

Thus, if $n_1 \geq 4$, total revenue after the merger is lower.

Our next result generalizes the above observation to any distribution, as long as the number of buyers is sufficiently large.

Proposition 5 *Assume that two separate auction sites, each with one seller and $n_i = m$ buyers, merge, and assume that the two sellers subsequently compete in a unique auction with $2m$ buyers. If m is large enough, total revenue goes down.*

4   Competition among auction sites

We now consider two competing, uniform price, multi-unit auctions. The two sellers simultaneously choose the number of units for sale k_i, $i = 1, 2$ in their respective auctions. The buyers observe the respective supplies at each site, and then decide where to bid, and what bid to place. Since each auction is uniform, the bidding part of the decision is simple, and of no further concern: each bidders places a bid equal to his valuation. Assume then that n_1 buyers compete in auction 1 for k_1 units, while n_2 buyers compete in auction 2 for k_2 units, where $k_1 + k_2 < n = n_1 + n_2$.

Definition 2 *A profile $(n_1^*, k_1^*; n_2^*, k_2^*)$ is an equilibrium of the competing auctions model if :*

(i) k_i^* maximizes seller $i's$ revenue given $k_{-i.}^*, n_1^*, n_2^*$. That is, no seller wishes to adjust supply.

(ii) Given (k_1^*, k_2^*) and given all other buyers' decisions, no buyer h has an incentive to switch to another auction site.

Note that in an equilibrium with positive profits at two auction sites we cannot have $k_1^* + k_2^* = n$. In that case, the supply at one auction site (at least) is no less than the number of buyers , and the respective price is then zero.

The unit price in auction i is given by $EX_{n_i^* - k_i^*, n_i^*}$. The following Lemma gives a condition on the prices at the two sites that must hold in equilibrium:

Lemma 1 *Consider any fixed strategy profile for the two sellers and for all buyers except h. Then, it is optimal for h to join the auction with the lower expected price. Moreover, in any equilibrium, $(n_1^*, k_1^*; n_2^*, k_2^*)$ it must hold that*

$$EX_{n_i^*-k_i^*, n_i^*} \geq EX_{n_j^*-k_j^*, n_j^*} \Rightarrow EX_{n_i^*-k_i^*, n_i^*} \leq EX_{n_j^*+1-k_j^*, n_j^*+1}$$

In other words, in equilibrium, it cannot be profitable for a buyer in the auction with a higher expected price to move to the other auction. Thus, while equilibrium prices at the two auction sites need not be strictly equal, there are no arbitrage opportunities.

4.1 Equilibrium coexistence of two auction sites

Our main result in this section is that two auction sites yielding positive profits for the respective sellers cannot coexist in equilibrium if the distribution of buyers' values is IFR, and if the marginal production cost, c, is sufficiently low. The proof is based on the following Lemma which has some independent interest. It shows that, for any profile of buyers' actions, at least one of the sellers has an incentive to increase supply if F is IFR.

Lemma 2   *1. Consider any configuration $(n_1, k_1; n_2, k_2)$ where $n_1, n_2 > 0$, $k_1, k_2 > 0$ and $k_1 + k_2 < n$, and assume that buyers play a best response to any announcement of quantities. If $EX_{n_i-k_i, n_i} \leq EX_{n_j-k_j, n_j}$, $i, j = 1, 2$, then seller i can attract at least one more buyer by supplying one additional unit.*

   2. Assume that F is IFR, and that $c = 0$. For any configuration $(n_1, k_1; n_2, k_2)$ where $n_1, n_2 > 0$, $k_1, k_2 > 0$ and $k_1 + k_2 < n$, one of the sellers can increase his profit by supplying one additional unit.

   3. If F is IFR, and if $c = 0$, then $\Delta R(k, n) = R(k+1, n+1) - R(k, n)$, the marginal revenue of seller whose supply increases from k to $k+1$ while the number of buyers attending his auction increases from n to $n+1$, decreases in k for a fixed n.

The Lemma yields:

Theorem 3 *Assume that F is IFR. Then, for sufficiently small marginal production costs c, there is no equilibrium in which both sellers are active and make positive profits. In particular, if*

$$c < \frac{n}{2} EX_{1, \frac{n}{2}+1} - (\frac{n}{2} - 1) EX_{1, \frac{n}{2}} \tag{5}$$

there is no symmetric equilibrium with positive profits.

If the IFR condition is not satisfied, or if the cost is high enough (in contrast to the assumptions of the above Proposition) an equilibrium where two auction sites are active, and where both sellers

make positive profits may then exist. We look first at an example where F does not satisfy the IFR condition:

Example 4 *Let $F(x) = x^{1/2}$, and let $c = 0$. Then:*

$$\Delta R\left(k_i, n_i\right) = \frac{(k_i + 1)(n_i - k_i + 1)(n_i - k_i)}{(n_i + 2)(n_i + 3)} - \frac{k_i(n_i - k_i + 1)(n_i - k_i)}{(n_i + 1)(n_i + 2)}$$

$$= \frac{(n_i - k_i)(n_i - k_i + 1)(n_i + 1 - 2k_i)}{(n_i + 1)(n_i + 2)(n_i + 3)}$$

Note that $\Delta R\left(k_i, n_i\right) \leq 0$ if $k_i \geq \frac{n_i+1}{2}$. Letting $k_1 = k_2 = 2$ and $n_1 = n_2 = 3$, it is clear that no seller has an incentive to increase supply . Assume then that seller 1, say, lowers supply to one unit. Then a buyer attending his auction has an incentive to switch to the other auction since there the price will be $EX_{2,4} < EX_{2,3}$. Therefore no seller has an incentive to deviate, and we obtain a symmetric equilibrium in which every seller sells two units and makes a positive profit.[13]

Next we look at situations where the marginal cost c is sufficiently high, Then, sellers do not want to increase their supply above a certain level (since the price drops below cost), and an equilibrium may exist.

If, in a symmetric equilibrium, a seller sells k^* units, he should have no incentive to increase supply to $k^* + 1$. Then, for F which is IFR, we have the necessary condition[14]

$$p = EX_{\frac{n}{2}-k^*, \frac{n}{2}} > c > (k^* + 1)EX_{\frac{n}{2}-k^*, \frac{n}{2}+1} - k^* EX_{\frac{n}{2}-k^*, \frac{n}{2}} \tag{6}$$

where the left inequality indicates that a seller makes a positive profit by selling k^* units to half the number of buyers, and the right inequality indicates that a seller has no incentive to increase the number of supplied units from k^* to $k^* + 1$ if by doing so the number of buyers attending his auction increases from $\frac{n}{2}$ to $\frac{n}{2} + 1$.

The following proposition provides a sufficient condition for existence of a symmetric equilibrium with the maximal supply value for which both sellers make positive profits.

Proposition 6 *Assume that F is IFR. If*

$$(\frac{n}{2} - 1)EX_{(2, \frac{n}{2}+1)} - (\frac{n}{2} - 2)EX_{(2, \frac{n}{2})} > c > \frac{n}{2}EX_{(1, \frac{n}{2}+1)} - (\frac{n}{2} - 1)EX_{(1, \frac{n}{2})} \tag{7}$$

then the profile where each seller sells $\frac{n}{2} - 1$ objects to $\frac{n}{2}$ buyers constitutes a symmetric equilibrium.

4.2 A unique auction site as the outcome of competition

We now focus on equilibria where only one auction site is active, and where only one seller makes a positive profit. Roughly speaking, such an equilibrium may exist if the optimal monopolistic quantity

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[13] In this example there is also a symmetric equilibrium where each seller supplies three units, and where profits are zero.

[14] If F is IFR, by lemma 2 the RHS of (6) is always positive.

for the unique active seller is high enough. Then, the other seller cannot convince enough buyers to switch to his own auction, and thus prefers to stay out.

Formally, an equilibrium with the form $(n_1^* = n, \; k_1^* > 0, n_2^* = 0, k_2^* = 0)$ exists if and only if:

1.

$$k_1^* = \arg\max_k k(EX_{n-k,n} - c) > 0 \tag{8}$$

and

2. For all $k_2 \leq n - k_1^*$, either the set $M = M(k_2) = \{m \mid m \leq n - k_1^* \wedge EX_{m-k_2,m} \geq c\}$ is empty or

$$EX_{\widetilde{n}_2 - k_2, \widetilde{n}_2} > EX_{n - \widetilde{n}_2 + 1 - k_1^*, n - \widetilde{n}_2 + 1} \tag{9}$$

where $\widetilde{n}_2 = \widetilde{n}_2(k_2)$ is the minimal element in M.

Condition (8) requires that seller 1 optimizes his (monopolistic) supply and makes a positive profit, given that seller 2 is not active. In condition (9), $\widetilde{n}_2$ is the minimal number of buyers for which selling k_2 units is profitable for seller 2. Note that if condition (9) holds and if $n_2 > \widetilde{n}_2$, we also get

$$EX_{n_2 - k_2, n_2} > EX_{\widetilde{n}_2 - k_2, \widetilde{n}_2} > EX_{n - \widetilde{n}_2 + 1 - k_1^*, n - \widetilde{n}_2 + 1} = EX_{n - n_2 + (n_2 - \widetilde{n}_2 + 1) - k_1^*, n - n_2 + (n_2 - \widetilde{n}_2 + 1)}$$

The first inequality follows by repeated application of the known relation $EX_{(i+1,n+1)} \geq EX_{(i,n)}$, and the second inequality is condition (9). That is, condition (9) guarantees that if $n_2 > \widetilde{n}_2$ buyers attend auction 2, at least $(n_2 - \widetilde{n}_2 + 1)$ buyers will want to switch to auction 1. Thus, seller 2 must remain inactive, and earns zero profit.

Proposition 7 *1. Assume that $k_1^* = \arg\max_k k(EX_{n-k,n} - c)$ is sufficiently large (For example, assume that F is sufficiently convex and that c is sufficiently small). Then there exists an asymmetric equilibrium with a unique active seller.*

2. Assume that $k_1^* = \arg\max_k k(EX_{n-k,n} - c) < \frac{n}{2}$. (For example, assume that F is sufficiently concave) Then there exists no equilibrium with a unique active seller.

The following examples illustrates the result.

Example 5 *Let $c = 0$, $n = 4$ and let F be uniform on $[0,1]$. By Corollary 2 the optimal supply for a monopolistic seller 1 is $k_1^* = 2$. The only alternative for seller 2 that leads to a positive profit is to supply one unit. Since $EX_{1,2} - EX_{4-1-2,4-1} = EX_{1,2} - EX_{1,3} > 0$, condition (9) is satisfied. Seller 2 cannot attract more than one buyer to his auction, and he therefore prefers to stay out.*

5 Comparison among models

How do the models analyzed in this paper (and in particular the competition models with either one or two auctions sites) compare in terms of equilibrium prices and quantities ? In this section we answer this question under the assumption that the number of buyers is large. We consider the following:

- Monopoly (M): one auction site with one seller and n buyers;

- Competition among sellers at one auction site (C1): two competing sellers at one auction site with n buyers;

- Competing auctions (C2): two sellers running separate auctions, competing to attract n buyers. For the results below we assume that a symmetric equilibrium exists in this model.

Proposition 8 *Assume that F, the distribution of buyers' values, is IFR. Assume total equilibrium supply in all three models is at least $\frac{n}{3} + 2$. (This happens, for example, if either F is convex enough with respect to the exponential distribution, or if the marginal cost is not too high). Then the following hold:*

(1) The equilibrium price under monopoly is higher than the prices in symmetric equilibria of the competition models.

(2) If total supply is higher (lower) than $\frac{n}{2} + 2$, then the equilibrium price in the competition model with one auction site is higher (lower) than the equilibrium price in the competition model with two auction sites.

6 Concluding remarks

We have studied competition among two sellers who optimize their respective supply in markets with a finite, not necessarily large, number of buyers. We studied a model where all transactions take place at price determined in one auction, and another where there two separate auction sites may operate side by side. A unified perspective on both models can be obtained by considering buyers' switching costs. Whereas an equilibrium where both sellers make profits always exists in the first model (one auction site), this is not the case in the second model (two auction sites) under ubiquitous conditions on the demand function (decreasing marginal revenue) and on the supply functions (sufficiently low marginal costs). In markets with sinking production (or trading) costs, our results suggest a movement towards dominant market places featuring several competing suppliers. Ebay is an obvious example. Instead of a competition mode characterized by poaching business from competitors, we currently observe a move towards consolidation also in the trading of securities.[15] For example, on October 17th 2006 the Chicago Mercantile Exchange agreed to buy the Chicago Board of Trade, thus aiming to create the

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[15] See "The Big Squeeze", *The Economist*, October 7th, 2006, pp. 83.

world's biggest financial marketplace.[16] Similarly, the Deutsche Börse has long argued for the need of a pan-European exchange, and tried (unsuccessfully) to buy the London Stock Exchange – which is now on NASDAQ's buying list.

Finally, we hope that the mathematical methods used in this paper will prove to be useful in a variety of other competition models with a small number of traders.

7 Appendix A

Barlow and Porschan (1966) proved the following 5 results about order statistics:

Lemma 3 *Assume that $F(0) = 0$. If F is convex (concave) then $(n+1)EX_{i,n}/i$ is decreasing (increasing) in i and increasing (decreasing) in n.*

Lemma 4 *Let $G^{-1}F$ be star-shaped on the support of F and assume $F(0) = G(0) = 0$. Then EX_{in}/EY_{in} is decreasing in i.*

Lemma 5 *If F is $IFR(DFR)$ and if $F(0) = 0$, then $\sum_1^r (n-i+1)(X_{i,n} - X_{i-1,n})$ is stochastically increasing (decreasing) in $n \geq r$.*

Lemma 6 *If F is IFR (DFR) and if $F(0) = 0$, then $(n-i+1)(X_{i,n} - X_{i-1,n})$ is stochastically increasing (decreasing) in $n \geq i$ for fixed i.*

Lemma 7 *If F is IFR (DFR) and if $F(0) = 0$, then $(n-i+1)(X_{i,n} - X_{i-1,n})$ is stochastically decreasing (decreasing) in $i \leq n$ for fixed n.*

8 Appendix B

Proof of Proposition 1:

If the distribution of buyers' valuations is F, the seller's revenue when he sells k units is

$$kEX_{n-k,n} = (n-r)EX_{r,n} \qquad \text{where } r \equiv n - k$$

Suppose that $r^* = n - k_F^*(n)$ is optimal: selling $k_F^*(n)$ is always better than selling $k' > k_F^*(n)$. Letting $r' = n - k' < r^*$, we obtain

$$(n-r^*)EX_{r^*,n} \geq (n-r')EX_{r',n} \Leftrightarrow \frac{EX_{r^*,n}}{EX_{r',n}} \geq \frac{n-r'}{n-r^*} \tag{10}$$

For distribution is G, the seller's revenue when he sells k units is

$$(n-k)\,EY_{n-k,n} = (n-r)\,EY_{r,n} \quad \text{where } r \equiv n-k$$

Suppose that $k_G^*(n)$ units are now optimal. In order to show that $k_G^*(n) \leq k_F^*(n)$, it is sufficient to show that under distribution G, the seller's revenue of selling $k_F^*(n)$ units is always larger than when selling $k' > k_F^*(n)$. That is, we need to show that, for $r^* > r'$

$$(n-r^*)\,EY_{r^*,n} \geq (n-r')\,EY_{r',n} \Leftrightarrow \frac{EY_{r^*,n}}{EY_{r',n}} \geq \frac{n-r'}{n-r^*}$$

Given condition (10), it is sufficient to show that

$$\frac{EY_{r^*,n}}{EY_{r',n}} \geq \frac{EX_{r^*,n}}{EX_{r',n}} \Leftrightarrow \frac{EX_{r^*,n}}{EY_{r^*,n}} \leq \frac{EX_{r',n}}{EY_{r',n}}$$

By Lemma 4 in Appendix A, EX_{in}/EY_{in} is decreasing in i . Since $r^* > r'$, the last inequality holds. **Q.E.D.**

Proof of Corollary 1:

When n is large, the optimal number of objects $k_G^*(n)$ for the exponential distribution G is

$$
\begin{aligned}
k_G^*(n) &\in \arg\max_k kEX_{n-k,n} \approx \arg\max_k kG^{-1}\left[\frac{n-k}{n}\right] = \arg\max\left(k\ln\frac{n}{k}\right) \\
&\Rightarrow k_G^*(n) \approx \frac{n}{e}
\end{aligned}
$$

$F(0) = G(0) = 0$ and $G^{-1}F$ convex imply together that $G^{-1}F$ star-shaped. If G is the exponential distribution, $G^{-1}F$ convex (where finite) is equivalent to F being IFR. Thus, applying proposition 1, we obtain that $k_F^*(n) \geq n/e$ if F is IFR. The reverse inequality holds for F which is DFR. **Q.E.D.**

Proof of Corollary 2:

If G is uniform distribution, the optimal number of objects $k_G^*(n)$ is

$$
\begin{aligned}
k_G^*(n) &\in \arg\max kEX_{n-k,n} = \arg\max \frac{k(n-k)}{n+1} \\
&\Rightarrow k_G^*(n) = \frac{n}{2}
\end{aligned}
$$

Applying proposition 1, we obtain that $k_F^*(n) \geq (\leq) \frac{n}{2}$ if F is convex (concave) **Q.E.D.**

Proof of Proposition 2:

By Lemma 6 in Appendix A, if F is $IFR(DFR)$, then $(n-i+1)(EX_{i,n} - EX_{i-1,n})$ is stochastically increasing (decreasing) in $n \geq i$ for fixed i. Thus, if F is IFR we obtain:

$$(n-i+2)(EX_{i,n+1} - EX_{i-1,n+1}) > (n-i+1)(EX_{i,n} - EX_{i-1,n})$$

This implies that

$$(n - i + 1)EX_{i,n+1} - (n - i + 2)EX_{i-1,n+1} + EX_{i,n+1} > (n - i)EX_{i,n} - (n - i + 1)EX_{i-1,n} + EX_{i,n}$$

Since $EX_{i,n+1} < EX_{i,n}$ we obtain

$$(n - i + 1)EX_{i,n+1} - (n - i + 2)EX_{i-1,n+1} > (n - i)EX_{i,n} - (n - i + 1)EX_{i-1,n}$$

Letting $i = n - k$ in the inequality yields

$$(k + 1)EX_{n+1-(k+1),n+1} - (k + 2)EX_{n+1-(k+2),n+1} > kEX_{n-k+,n} - (k + 1)EX_{n-(k+1),n}$$

Thus, if it is profitable to decrease the number of objects from $k + 1$ to k when the number of buyers is n, then it is also profitable to decrease the number of objects from $k + 2$ to $k + 1$ when the number of buyers is $n + 1$. In particular, $k_F^*(n + 1) \leq k_F^*(n) + 1$. **Q.E.D.**

Proof of Proposition 3:

By Lemma 7 in Appendix A, if F is IFR, then $(n - i + 1)(EX_{i,n} - EX_{i-1,n})$ is stochastically decreasing in $i \leq n$ for fixed n. Thus, if F is IFR we obtain:

$$(n - i + 1)(EX_{i,n} - EX_{i-1,n}) > (n - (i + 1) + 1)(EX_{i+1,n} - EX_{i,n})$$

Letting $k = n - i$ in the the last inequality yields:

$$kEX_{n-k,n} - (k + 1)EX_{n-(k+1),n} + EX_{n-k,n} > (k - 1)EX_{n-(k-1),n} - kEX_{n-k,n} + EX_{n-(k-1),n}$$

Since $EX_{n-k,n} < EX_{n-(k-1),n}$ we obtain

$$kEX_{n-k,n} - (k - 1)EX_{n-(k-1),n} > (k + 1)EX_{n-(k+1),n} - kEX_{n-k,n}$$

The last inequality indicates that the marginal revenue of a seller is decreasing in the number of objects and this property implies the desired result. **Q.E.D.**

Derivations for Example 2: **1)** If $n = 3m$, where m is an integer, inequality 4 becomes,

$$m - 1 \leq k_1 \leq m + 1 \Rightarrow k_1 = m - 1, \ m, \ \text{or} \ m + 1$$

Suppose $k_1^* = m - 1$, then

$$\pi_2 = \frac{k_2 (n - k_1 - k_2)}{n + 1} = \frac{k_2 (2m + 1 - k_2)}{3m + 1} \Rightarrow k_2^* = m, \ \text{or} \ k_2^* = m + 1$$

But if $k_2^* = m$, the best response of seller 1 is $k_1 = m$. Therefore, $k_1^* = m - 1$ and $k_2^* = m$ is not an equilibrium profile. If $k_2^* = m + 1$, the best response for firm 1 is $k_1 = m$, or $m - 1$. Thus, there are

three possible pure strategy equilibria

$$k_1 = m+1, k_2 = m-1$$
$$k_1 = m-1, k_2 = m+1$$
$$k_1 = k_2 = m$$

2) If $n = 3m+1$, where m is an integer, inequality 4 becomes,

$$m - \frac{2}{3} \leq k_1 \leq m + \frac{4}{3} \Rightarrow k_1 = m, \text{ or } m+1$$

Suppose $k_1^* = m$, then

$$\pi_2 = \frac{k_2 (n - k_1 - k_2)}{n+1} = \frac{k_2 (3m+1-m-k_2)}{n+1} = \frac{k_2 (2m+1-k_2)}{3m+2} \Rightarrow k_2^* = m \text{ or } m+1$$

Suppose $k_1^* = m+1$, then

$$\pi_2 = \frac{k_2 (n - k_1 - k_2)}{n+1} = \frac{k_2 (3m+1-m-1-k_2)}{n+1} = \frac{k_2 (2m-k_2)}{3m+2} \Rightarrow k_2^* = m$$

Again, there will be three pure-strategy equilibria

$$k_1 = m+1, \ k_2 = m$$
$$k_1 = m, \ k_2 = m+1$$
$$k_1 = k_2 = m$$

3) If $n = 3m+2$, where m is an integer, inequality 4 becomes

$$m - \frac{1}{3} \leq k_1 \leq m + \frac{5}{3} \Rightarrow k_1 = m, \text{ or } m+1$$

Suppose $k_1^* = m$, then

$$\pi_2 = \frac{k_2 (n - k_1 - k_2)}{n+1} = \frac{k_2 (3m+2-m-k_2)}{n+1} = \frac{k_2 (2m+2-k_2)}{3m+2} \Rightarrow k_2^* = m+1$$

Therefore, there will be two asymmetric pure strategy equilibria

$$k_1 = m, \ k_2 = m+1$$
$$k_1 = m+1, \ k_2 = m$$

Q.E.D.

Proof of Theorem 1: Since profits are zero if the produced quantity is larger than the number of bidders, we can assume without loss of generality that the sellers choose quantities out of a finite set of integers: $\Omega = \{0, 1, 2, ..., K\}$, $K < \infty$. Given k_2, seller 1 chooses $k_1 \in \Omega$ to maximize the payoff function

$$R_1 (k_1, k_2) = k_1 E X_{n-(k_1+k_2),n}.$$

and given k_1, seller 2 chooses $k_2 \in \Omega$ to maximize the payoff function

$$R_2\left(k_1, k_2\right) = k_2 EX_{n-(k_1+k_2),n}$$

Let Γ denote the above described two-person game, and define

$$P\left(k_1, k_2\right) = k_1 k_2 EX_{n-(k_1+k_2),n}$$

We now verify that P is an ordinal potential of Γ.[17] We need to check that:

$$\forall k_2 \;>\; 0, \quad R_1\left(k_1, k_2\right) - R_1\left(k_1', k_2\right) > 0 \Leftrightarrow P\left(k_1, k_2\right) - P\left(k_1', k_2\right) > 0$$

$$\forall k_1 \;>\; 0, \quad R_2\left(k_1, k_2\right) - R_2\left(k_1, k_2'\right) > 0 \Leftrightarrow P\left(k_1, k_2\right) - P\left(k_1, k_2'\right) > 0$$

The above conditions become

$$k_1 EX_{n-(k_1+k_2),n} - k_1' EX_{n-\left(k_1'+k_2\right),n} \;>\; 0 \Leftrightarrow k_1 k_2 EX_{n-(k_1+k_2),n} - k_1' k_2 EX_{n-\left(k_1'+k_2\right),n} > 0$$

$$k_2 EX_{n-(k_1+k_2),n} - k_2' EX_{n-\left(k_1'+k_2\right),n} \;>\; 0 \Leftrightarrow k_1 k_2 EX_{n-(k_1+k_2),n} - k_1 k_2' EX_{n-\left(k_1'+k_2\right),n} > 0$$

which trivially hold.

Since both k_1 and k_2 are chosen from a finite set Ω, it is obvious that the potential P has a maximum on $\Omega \times \Omega$. (Note that the maximum need not be unique). The existence result follows now directly from the following result:

Lemma 8 (Lemma 2.1, Monderer and Shapley, 1996) *Let P be an ordinal potential function for Γ. Then the equilibrium set of $\Gamma\left(R_1, R_2\right)$ coincides with the equilibrium set of $\Gamma\left(P, P\right)$. That is, $\left(k_1, k_2\right) \in \Omega \times \Omega$ is an equilibrium point for Γ if and only if for every $i \in \{1, 2\}$,*

$$P\left(k_i, k_{-i}\right) \geq P\left(k_i', k_{-i}\right) \quad \text{for every } k_i' \in \Omega.$$

Consequently, if P admits a maximal value in $\Omega \times \Omega$, then $\Gamma\left(R_1, R_2\right)$ possesses a pure strategy equilibrium.

Q.E.D.

Proof of Theorem 2: By the above proof, the potential P has a maximum. Suppose that this maximum is achieved at $\left(k_1^*, k_2^*\right)$. We need to show that $\left|k_1^* - k_2^*\right| \leq 1$. Assume, by contradiction, that the opposite holds:

$$\left|k_1^* - k_2^*\right| \;>\; 1$$

$$\Leftrightarrow \;\left|k_1^* - k_2^*\right| \geq 2$$

---

[17] If the firms face a marginal cost $c > 0$, define the potential $P\left(k_1, k_2\right) = k_1 k_2 (EX_{n-(k_1+k_2),n} - c)$. This and the next proof proceed then in an analogous fashion yielding the respective results.

Without loss of generality assume that

$$k_1^* \geq k_2^* + 2. \tag{11}$$

Consider now (k_1', k_2') in $\Omega \times \Omega$ defined by

$$
\begin{aligned}
k_1' &= k_1^* - 1 \\
k_2' &= k_2^* + 1
\end{aligned}
$$

and observe that $k_1' + k_2' = k_1^* + k_2^*$. We obtain then that

$$
\begin{aligned}
&P(k_1', k_2') - P(k_1^*, k_2^*) \\
&= (k_1^* - 1)(k_2^* + 1) EX_{n-(k_1^*+k_2^*),n} - k_1^* k_2^* EX_{n-(k_1^*+k_2^*),n} \\
&= (k_1^* - k_2^* - 1) EX_{n-(k_1^*+k_2^*),n} > 0
\end{aligned}
$$

where the last inequality follows by (11). This yields a contradiction to the assumption that P achieves a maximum at (k_1^*, k_2^*). Thus, $|k_1^* - k_2^*| \leq 1$ as desired. **Q.E.D.**

Proof of Proposition 4:

1) Suppose F is convex. Given k_2, seller 1 chooses k_1 to maximize

$$R_1 = k_1 EX_{(n-k_2)-k_1,n}$$

By Corollary 2, if F is convex, $k_1 \geq \frac{1}{2}(n - k_2)$. Analogously, we have $k_2 \geq \frac{1}{2}(n - k_1)$. Combining the two inequalities, it follows that

$$k_1 + k_2 \geq \frac{2}{3}n$$

The opposite inequality holds for concave F.

2) Now suppose F is IFR and n is large. Given k_2, seller 1 chooses k_1 to maximize

$$R_1 = k_1 EX_{(n-k_2)-k_1,n}$$

By Corollary 1, we have $k_1 \geq \frac{1}{e}(n - k_2)$. Similarly, we have $k_2 \geq \frac{1}{e}(n - k_1)$. Therefore,

$$k_1 + k_2 \geq \frac{2}{1+e}n$$

The proof for the DFR case is analogous. **Q.E.D.**

Proof of Proposition 5: Suppose that $m = n_1 = n_2$ is large enough. Total revenue before the merger is

$$\pi_1 + \pi_2 = 2 \max_k k F^{-1}\left(\frac{m-k}{m}\right)$$

Let $k^* \in \arg\max k F^{-1}\left(\frac{m-k}{m}\right)$.

In the post merger competition the revenues are given by:

$$\pi_1^c = \pi_2^c = \max_{k_i} k_i F^{-1}\left(\frac{2m - k_i - k_{-i}}{2m}\right)$$

We focus on the (quasi) symmetric equilibrium in the post merger games (which always exists by Proposition 2). The potential function is

$$P(k_1, k_2) = k_1 k_2 F^{-1}\left(\frac{2m - k_1 - k_2}{2m}\right)$$

Since $|k_1^c - k_2^c| \leq 1$, the profile $(k_1^c, \; k_2^c)$ that maximizes the potential must satisfy

$$k_i^c \in \arg\max (k)^2 \, F^{-1}\left(\frac{m - k}{m}\right), \; i = 1, 2$$

One implication is that

$$k_1^c \notin \arg\max k F^{-1}\left(\frac{m - k}{m}\right)$$

The change in total revenue is:

$$\begin{aligned}
&\pi_1^c + \pi_2^c - (\pi_1 + \pi_2) \\
=\; & 2k_1^c F^{-1}\left(\frac{m - k_1^c}{m}\right) - 2k^* F^{-1}\left(\frac{m - k^*}{m}\right) \\
<\; & 0
\end{aligned}$$

because $k^* \in \arg\max k F^{-1}\left(\frac{m-k}{m}\right)$. Thus, a merger reduces total revenue. **Q.E.D.**

Proof of Lemma 1: Fix a strategy profile for all agents except buyer h, and assume that in this profile the respective quantities and number of bidders at each site are k_1, k_2, n_1, n_2, where $n_1 + n_2 = n - 1$. Denote by X the random variable representing bidder h's valuation. If h joins site $i, i = 1, 2$, his expected payoff is given by

$$\Pr\{X \geq EX_{n_i+1-k_i,n_i+1}\}E[X - EX_{n_i+1-k_i,n_i+1} \mid X \geq EX_{n_i+1-k_i,n_i+1}], \; i = 1, 2$$

It is immediate that the above expression is higher for $i = \arg\min \{EX_{n_1+1-k_1,n_1+1}, EX_{n_2+1-k_2,n_2+1}\}$, thus it is optimal for bidder h to join the site with the lower expected price. The second argument follows then immediately by the definition of equilibrium. **Q.E.D.**

Proof of Lemma 2:

1) Suppose without loss of generality that $EX_{n_1-k_1,n_1} \leq EX_{n_2-k_2,n_2}$. Since

$$EX_{n_1+1-(k_1+1),n_1+1} = EX_{n_1-k_1,n_1+1} < EX_{n_1-k_1,n_1}$$

we obtain

$$EX_{n_1+1-(k_1+1),n_1+1} < EX_{n_2-k_2,n_2}$$

The last inequality provides a sufficient condition for a buyer to move from auction 2 to auction 1 if seller 1 increases the supplied quantity by one unit while all other agents stay put.

2) Let $n_1 = m$. By Lemma 5 in Appendix A, if F is IFR then $\sum_1^r (m - i + 1)(EX_{i,m} - EX_{i-1,m})$ is stochastically increasing in $m \geq r$. Let $r = 1$, then

$$mEX_{1,m} < (m+1)EX_{1,m+1}$$

Since $EX_{1,m+1} < EX_{1,m}$, we have

$$(m-1)EX_{1,m} < mEX_{1,m+1}$$

Similarly, by Lemma 5

$$EX_{1,m} + (m-1)EX_{2,m} < EX_{1,m+1} + mEX_{2,m+1}$$

Since $EX_{2,m+1} < EX_{2,m}$, we have

$$(m-2)EX_{2,m} < (m-1)EX_{2,m+1}$$

By induction, for all $1 \leq i < m$,

$$(m-i)EX_{i,m} < (m-i+1)EX_{i,m+1}$$

Letting $i = m - k$, we get

$$kEX_{m-k,m} < (k+1)EX_{m-k,m+1}$$

The last inequality implies that a seller increases his revenue by attracting one more buyer from the other site.

3) Suppose that the number of buyers in the auction is m. By Lemma 6, if F is $IFR(DFR)$ then $(m - i + 1)(EX_{i,m} - EX_{i-1,m})$ is stochastically increasing (decreasing) in $m \geq i$ for fixed i. Thus,

$$(m-i+2)(EX_{i,m+1} - EX_{i-1,m+1}) > (m-i+1)(EX_{i,m} - EX_{i-1,m})$$

This implies that

$$(m-i+2)EX_{i,m+1} - (m-i+1)EX_{i,m} > (m-i+2)EX_{i-1,m+1} - (m-i+1)EX_{i-1,m}$$

Since $EX_{i,m+1} < EX_{i,m}$ we obtain

$$(m-i+1)EX_{i,m+1} - (m-i)EX_{i,m} > (m-i+2)EX_{i-1,m+1} - (m-i+1)EX_{i-1,m}$$

Letting $i = m - k$, we get

$$(k+1)EX_{m-k,m+1} - kEX_{m-k,m} > (k+2)EX_{m-k-1,m+1} - (k+1)EX_{m-k-1,m}$$

The last inequality implies that the marginal revenue of a seller that attracts one more buyer by supplying one more unit is decreasing in k. **Q.E.D.**

Proof of Theorem 3:

As mentioned in the text, if both sellers make a profit in equilibrium, we must have $k_1^* + k_2^* < n$. Assume that

$$c \leq \min_{s \in [1,n-1]; \ k \in [0,s-1]} \{(k+1)EX_{s-k,s+1} - kEX_{s-k,s}\}$$

Then, no matter how many units are supplied, and how many bidders attend, it is always profitable for a seller to increase supply by one unit if he attracts one more buyer. By Lemma 2-3, we know that marginal revenue is decreasing if F is IFR. Then, for each s, the minimum in the above condition is achieved for $k = s - 1$, and it is sufficient to assume that

$$c \leq \min_{s \in [1,n-1];} \{sEX_{1,s+1} - (s-1)EX_{1,s}\}$$

Suppose now, by contradiction, that $(n_1^*, k_1^*; n_2^*, k_2^*)$ is an equilibrium, where $n_1^*, n_2^* > 0$, $n_1^* + n_2^* = n$, $k_1^*, k_2^* > 0$ and $k_1^* + k_2^* < n$. Suppose also, without loss of generality, that seller 1 faces the lower expected auction price. Then this seller can increase the number of supplied units to $(k_1^* + 1)$. By Lemma 2, at least one buyer will want to switch to his auction, and, by the assumption on the marginal cost c, this constitutes a profitable deviation. More buyers switching is even more profitable since the price further increases. Therefore, $(n_1^*, k_1^*; n_2^*, k_2^*)$ cannot be an equilibrium.

Lemma 2 and condition (5) imply that a seller selling $k_i \leq \frac{n}{2} - 1$ objects to $\frac{n}{2}$ bidders has incentive to increase his supply. But, if both sellers together sell n objects, they cannot both make positive profits. **Q.E.D.**

Proof of Proposition 6:

Under condition (7) both sellers have an incentive to increase supply from $\frac{n}{2} - 2$ to $\frac{n}{2} - 1$, but no seller has an incentive to increase supply from $\frac{n}{2} - 1$ to $\frac{n}{2}$ if by doing so only one more buyer will be attracted. If a seller sells $\frac{n}{2} - 1$ units, at least $\frac{n}{2} - 1$ buyers will attend his auction. Thus, no seller can attract more than one buyer by increasing the number of supplied units beyond $\frac{n}{2} - 1$. Finally, no seller has an incentive to decrease supply from $k_i^* = \frac{n}{2} - 1$ since a decrease by k units induces at least k buyers to switch to the other auction. **Q.E.D.**

Proof of Proposition 7:

1) Assume that seller 1 sets $k_1 = n - 1$. In that case, at least $n_1 = n - 1$ will always attend auction 1, and the price in auction 1 is at most $EX_{1,n}$. Seller 2 has no way of ensuring an equal or lower price. Thus, no matter what seller 2 does, he cannot attract enough buyers, and entry is not profitable.

Assume next that seller 1 sets $k_1 = n - 2$. Given that at least $n_1 = n - 2$ buyers always attend auction 1, the only way in which seller 2 can make a profit is by auctioning one unit to 2 buyers. In that case the price is $EX_{1,2}$. By switching to auction 1, one of these two buyers will face a price of $EX_{1,n-1} < EX_{1,2}$. Thus, switching is profitable, and seller 2 can never make a profit by entry.

Continuing in this way, let $\widetilde{k}_1$ be the minimal k_1 such that for all $k_2 < n - \widetilde{k}_1$, and for all n_2, $k_2 < n_2 < n - \widetilde{k}_1$, it holds that

$$EX_{n_2-k_2,n_2} > EX_{n-n_2+1-\widetilde{k}_1,n-n_2+1} > 0$$

Assume then that $k_1^* \geq \widetilde{k}_1$, and assume that selling k_1^* is profitable for seller 1 (condition (8)) .Then, the above inequality implies condition (9), and thus the existence of an equilibrium with a unique active site.

The assumptions in the statement are satisfied for sufficiently convex F and sufficiently small c by Proposition 1.

2) Assume that $k_1^* < \frac{n}{2}$. Setting $k_2 = k_1^*$, we obtain:

$$
\begin{aligned}
EX_{n_2-k_2,n_2} - EX_{n-n_2+1-k_1^*,n-n_2+1} &= EX_{n_2-k_1^*,n_2} - EX_{n-n_2+1-k_1^*,n-n_2+1} \\
&= EX_{n_2-k_1^*,n_2} - EX_{n_2-k_1^*+(n-2n_2+1),n_2+(n-2n_2+1)} < 0
\end{aligned}
$$

This follows because $n - 2n_2 + 1 > 0$ ($k_2 = k_1^*$ implies $n_1 = n_2$) and because $EX_{i,j} < EX_{i+a,j+a}$ for all $i < j$ and $a > 0$. In particular, condition (9) cannot be satisfied. Thus, by replicating seller 1's strategy, seller 2 can sustain a positive profit, and an asymmetric equilibrium with a unique active seller does not exist.

The assumption $k_1^* < \frac{n}{2}$ is clearly satisfied if F is sufficiently concave – see Proposition 1 and Corollary 2. **Q.E.D.**

Proof of Proposition 8: Assume that total supply in monopoly is $2k$ units. Denote by $g(x) = (F^{-1})'(x)$. Then, the marginal revenue obtained by supplying an additional unit is given by:

$$
\begin{aligned}
MR^M(k) &= (2k+1)EX_{n-(2k+1),n} - 2kEX_{n-2k,n} \\
&= EX_{n-(2k+1),n} - 2k\left(EX_{n-2k,n} - EX_{n-(2k+1),n}\right) \\
&\approx F^{-1}\left(\frac{n-2k-1}{n+1}\right) - 2k\left[F^{-1}\left(\frac{n-2k}{n+1}\right) - F^{-1}\left(\frac{n-2k-1}{n+1}\right)\right] \\
&\approx F^{-1}\left(\frac{n-2k-1}{n+1}\right) - \frac{2kg\left(\frac{n-2k}{n+1}\right)}{(n+1)}
\end{aligned}
$$

For the competition model with one auction site and total supply of $2k$, the marginal revenue of a

seller is:

$$
\begin{aligned}
MR^{C1}(k) &= (k+1)EX_{n-(2k+1),n} - kEX_{n-2k,n} \\
&= EX_{n-(2k+1),n} - k\left(EX_{n-2k,n} - EX_{n-(2k+1),n}\right) \\
&\approx F^{-1}\left(\frac{n-2k-1}{n+1}\right) - k\left[F^{-1}\left(\frac{n-2k}{n+1}\right) - F^{-1}\left(\frac{n-2k-1}{n+1}\right)\right] \\
&\approx F^{-1}\left(\frac{n-2k-1}{n+1}\right) - \frac{kg\left(\frac{n-2k}{n+1}\right)}{(n+1)}
\end{aligned}
$$

For the competition model with two sites and total supply $2k$, assume that a seller attracts s additional bidders by increasing supply by one unit. Then his marginal revenue is:

$$
MR^{C2}(k) = (k+1)EX_{\left(\frac{n}{2}+s\right)-(k+1),\,\frac{n}{2}+s} - kEX_{\frac{n}{2}-k,\,\frac{n}{2}}
$$

By the assumption that equilibrium supply is at least $\frac{n}{3}+2$ units, and by Lemma 9 below, we get that $s=1$. This yields:

$$
\begin{aligned}
MR^{C2}(k) &= (k+1)EX_{\frac{n}{2}-k,\,\frac{n}{2}+1} - kEX_{\frac{n}{2}-k,\,\frac{n}{2}} \\
&= EX_{\frac{n}{2}-k,\,\frac{n}{2}+1} - k\left(EX_{\frac{n}{2}-k,\,\frac{n}{2}} - EX_{\frac{n}{2}-k,\,\frac{n}{2}+1}\right) \\
&\approx F^{-1}\left(\frac{\frac{n}{2}-k}{\frac{n}{2}+2}\right) - k\left[F^{-1}\left(\frac{\frac{n}{2}-k}{\frac{n}{2}+1}\right) - F^{-1}\left(\frac{\frac{n}{2}-k}{\frac{n}{2}+2}\right)\right] \\
&\approx F^{-1}\left(\frac{n-2k}{n+4}\right) - kg\left(\frac{\frac{n}{2}-k}{\frac{n}{2}+1}\right)\left(\frac{\frac{n}{2}-k}{\frac{n}{2}+1} - \frac{\frac{n}{2}-k}{\frac{n}{2}+2}\right) \\
&= F^{-1}\left(\frac{n-2k}{n+4}\right) - \frac{kg\left(\frac{n-2k}{n+2}\right)(n-2k)}{(n+2)\left(\frac{n}{2}+2\right)}
\end{aligned}
$$

For large n we have

$$
\begin{aligned}
F^{-1}\left(\frac{n-2k-1}{n+1}\right) &\approx F^{-1}\left(\frac{n-2k}{n+4}\right) \\
\frac{kg\left(\frac{n-2k}{n+2}\right)}{(n+2)} &\approx \frac{kg\left(\frac{n-2k}{n+1}\right)}{(n+1)}
\end{aligned}
$$

Therefore, we obtain

$$
MR^{M}(k) < MR^{C1}(k) \text{ and } MR^{M}(k) < MR^{C2}(k) \text{ for all } k.^{[18]}
$$

On the other hand, note that

$$
\frac{(n-2k)}{\left(\frac{n}{2}+2\right)} > 1 \Leftrightarrow k < \frac{n}{4}+1.
$$

Thus, we have:

$$
\begin{cases}
MR^{C2}(k) < MR^{C1}(k) & \text{if } k < \frac{n}{4}+1 \\
MR^{C2}(k) > MR^{C1}(k) & \text{if } k > \frac{n}{4}+1
\end{cases}.
$$

---

[18] Note that the comparison among $MR^{M}(k)$ and $MR^{C1}(k)$ hold always, i.e., without any additional assumptions about the number of bidders, supply, etc..

If the distribution F has an increasing hazard rate, then by Lemma 2, the marginal revenue $MR^{C2}(k)$ is decreasing in k. Note also that

$$MR^M(k) = EX_{i,n} - (n-i+1)(EX_{i,n} - EX_{i-1,n}) \quad \text{where } i = n - 2k$$

$$MR^{C1}(k) = \frac{1}{2}EX_{i,n} + \frac{1}{2}EX_{i-1,n} - \frac{1}{2}(n-i+1)(EX_{i,n} - EX_{i-1,n}) \quad \text{where } i = n - 2k$$

By Lemma 7, both $MR^M(k)$ and $MR^{C1}(k)$ are decreasing in k if F is IFR.

Since the marginal cost is assumed to be the same across models, the equilibrium quantity is lowest and the price is the highest under monopoly. If marginal cost is relatively low, and assuming that an equilibrium exists in the competition model with two sites, this situation is more competitive than the one with one site. The opposite occurs for high marginal cost. **Q.E.D.**

When deriving the marginal revenue in the competition model with two auction sites, we assumed that the additional number of bidders that a seller can attract by increasing supply by one unit is one. The following lemma gives a mild condition under which this is indeed the case. It will be satisfied as long as c is not too large, such that the equilibrium supply of each seller satisfies $k > \frac{n}{6} + 1$. Analogous, weaker conditions can be derived whenever the number of attracted bidders is finite and small relative to the equilibrium quantity k.

Lemma 9 *Consider the competition model with two auction sites (C2) for a large number of buyers n. Suppose that the marginal cost c satisfies*

$$c < (\frac{n}{6} + 2)EX_{(\frac{n}{3}-1, \frac{n}{2}+1)} - \left(\frac{n}{6} + 1\right) EX_{(\frac{n}{3}-1, \frac{n}{2})} \tag{12}$$

Then, if a seller increases (decrease) supply by one unit, he attracts (looses) exactly one buyer;

Proof of Lemma 9: By the necessary condition for a symmetric equilibrium (6), and condition (12), we obtain that $k^* > \frac{n}{6} + 1$ where k^* denotes the equilibrium supply of each seller (otherwise, a seller always has incentive to increase quantity by one).

1) Suppose first that seller 1 increases supply by one unit. Lemma 2 shows that he can attract at least one more bidder. Therefore, we only need to show that he cannot attract two more bidders. That is, we need to show that

$$EX_{\frac{n}{2}+2-(k^*+1), \frac{n}{2}+2} > EX_{\frac{n}{2}-1-k^*, \frac{n}{2}-1}.$$

For large n, this is equivalent to showing

$$F^{-1}\left(\frac{\frac{n}{2}+1-k^*}{\frac{n}{2}+3}\right) > F^{-1}\left(\frac{\frac{n}{2}-1-k^*}{\frac{n}{2}}\right) \Leftrightarrow$$

$$\frac{\frac{n}{2}+1-k^*}{\frac{n}{2}+3} > \frac{\frac{n}{2}-1-k^*}{\frac{n}{2}} \Leftrightarrow$$

$$k^* > \frac{n}{6} + 1.$$

2) Suppose seller 1 decreases his supply by one. We show first that he will lose at least one bidder, and then we show that he will not lose more than one bidder.

(i) In order to show seller 1 will lose at least one bidder, it is sufficient to show

$$EX_{\frac{n}{2}-(k^*-1),\frac{n}{2}} > EX_{\frac{n}{2}+1-k^*,\frac{n}{2}+1}$$

which holds trivially.

(ii) In order show seller 1 will not lose more than one bidder, it is sufficient to show

$$EX_{\left(\frac{n}{2}-1\right)-(k^*-1),\frac{n}{2}-1} < EX_{\frac{n}{2}+2-k^*,\frac{n}{2}+2}.$$

For large n, this is equivalent to

$$
\begin{aligned}
F^{-1}\left(\frac{\frac{n}{2}-k^*}{\frac{n}{2}}\right) &< F^{-1}\left(\frac{\frac{n}{2}+2-k^*}{\frac{n}{2}+3}\right) \Leftrightarrow \\
\frac{\frac{n}{2}-k^*}{\frac{n}{2}} &< \frac{\frac{n}{2}+2-k^*}{\frac{n}{2}+3} \Leftrightarrow \\
k^* &> \frac{n}{6}.
\end{aligned}
$$

Q.E.D.

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