# Auctions with Downstream Interaction among Buyers

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**Abstract**

We study an auction whose outcome influences the future interaction among agents. The impact of that interaction on agent $i$ is assumed to be a function of all agents' types (which are private information at the time of the auction). Two explicit illustrations treat auctions of patents and takeover contests among oligopolists. We derive equilibrium bidding strategies for second-price, sealed-bid auctions in which the seller sometimes keeps the object, and we point out the various effects caused by positive and negative impacts. We also study the effect of reserve prices and entry fees on the seller's revenue and on welfare. We observe that these instruments have very different implications according to whether impacts are positive or negative.

## 1. Introduction

In a variety of settings significant changes of ownership influence the nature of the interaction in the respective markets. As a consequence, even agents who

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are not directly involved in a transaction are indirectly affected by its outcome. We refer to such indirect effects as *externalities*. Well-known examples involving externalities include[1]: 1) Changes of ownership in oligopolies (through takeover, merger, privatization, etc...): since the number and characteristics of active firms change, all firms operating in the industry will be affected (either negatively or positively) by the change of ownership; 2) The licensing of innovations (or the sale of intermediate inputs) to competing downstream producers: some firms become more efficient and downstream profits will be shifted away from other, relatively less efficient competitors. 3) The location of enterprises, which is often influenced by the award of tax rebates: neighboring communities may enjoy positive externalities on the labor market, but also negative externalities due to environmental hazards[2]. 4) A shareholder tendering his shares to a corporate raider creates a positive externality on other shareholders.

The anticipation of externalities often leads to adjustments in trading strategies. For an illustration, consider the following quotation from *The Economist,* June 28th, 1997:

> "The good sales run at Rolls-Royce began 18 months ago, when it snatched a huge order to supply Singapore Airlines with engines for its latest twin-engined Boeing 777s. Its hard-nosed American rivals, Pratt&Whitney and General Electric, were prepared to take a loss to land such a prestigious deal. So they assumed Rolls-Royce won the bid by taking an even greater loss."

The idea is that failing to get the prestigious Singapore order puts a firm in a disadvantageous position when bidding for later deals with other airlines. The need to avoid this disadvantage drove the competing firms to sacrifice profits on the current transaction[3].

[1] For more illustrations on bilateral contracts and externalities, see Segal, 1999.

[2] For example, at the end of the last century, the city of Mannheim asked a very high price for land and the newly created BASF moved westwards to Ludwigshafen, just across the bridge on the Rhine river. In the meanwhile, BASF became a chemical giant and it pays trade taxes (Gewerbesteuer) to the hosting municipality. But the wind blows eastward and the Rhine flows to the North sea...

[3] This phenomenon will be formalized in our discussion of auctions where failing to win has a negative impact on future expected profits.

In future deals with Singapore Airlines switching costs also play a role. Business strategies in the presence of switching costs are surveyed in Klemperer (1995).

Jehiel et. al. (1996, 1999) look at a model where one object is auctioned and where agents have private information about imposed or incurred externalities[4]. Their focus is on mechanism design[5] and on revenue maximizing sales procedures. By devising sophisticated threats which depend on the identities of the participating buyers[6], the seller can extract payments also from non-acquirers. A major problem with such procedures is that the seller needs an unrealistically strong commitment power[7].

In this paper we take a different approach by studying a standard second-price auction (whose specifications do not depend on the details of the underlying situation), and by focusing on simple revenue-enhancing instruments such as (fixed) reserve prices or entry fees. The second-price auction is chosen for its analytical simplicity: it allows us to highlight the interplay between allocative and informational interdependencies without getting too entangled in complex bidding mechanics. A similar analysis will hold for other sealed-bid mechanisms[8].

Besides allocative externalities, many applications require models that allow also for informational externalities, e.g., the externality on buyer $i$ depends both on $i$'s characteristics (which may be private information to $i$), and on characteristics of other agents, which are not observable at the transaction stage[9]

A classical symmetric one-object auction model allowing for informational interdependencies (but not for allocative ones) has been studied by Milgrom and Weber (1982). The main feature of that model is that bidder $i$'s valuation for the object is a function of the signals obtained by all bidders. Net of payments

[4] These depend on the identities of the actual buyer and the sufferer, but not on other characteristics

[5] The analysis employs and further develops the optimal mechanism design methodology for multi-dimensional type spaces.

[6] For example, personalized reserve prices and entry fees must be used. The seller may extract payments even if no exchange of goods occurs. Such a "chutzpah" mechanism has been derived in the licensing context by Kamien et.al. (1992).

[7] For example, the threats that allow extraction of surplus from non-acquirers will typically involve non-credible actions off the equilibrium path.

[8] Since types are independent a revenue equivalence theorem holds because in our symmetric environment all sealed bid auctions yield the same allocation. For the case of two bidders an English ascending auction is also equivalent to our mechanism, but this does not necessarily hold for more than two agents. An interesting comparison of revenue in sealed bid and ascending price auctions with externalities in which bidders'payoffs are determined by their own types in all alternatives (which is not the case here) is contained in Das Varma (1999).

[9] For example, information on the post-licensing cost of another firm in the innovation situation.

to the seller, an unsuccessful bidder obtains a fixed payoff, usually normalized to be zero[10]. In our model, the utility of a bidder who does not get the object is influenced by the realized allocation (e.g., by events such as "the good is not sold" , "the good is sold to another bidder"). Thus, bidder $i$'s willingness to pay depends on $i$'s belief about potential auction outcomes[11], and even in a complete information framework (see Jehiel and Moldovanu, 1996) bidding strategies are not trivial.

Allocative externalities have been often discussed in the large IO literature on vertical and horizontal relations. The type of analysis performed in this paper is strongly related to models considered in the literature on patent licensing (see the survey of Kamien,1992). Arrow (1962) discussed the relation between the value of innovations and the underlying market structure (which is assumed to be either competitive or monopolistic). Gilbert and Newbery (1982) use an auction model to study the interaction between a monopolist incumbent and a potential entrant competing for an innovation. Their main result is the persistence of the monopolist which takes into account the potential negative externality and uses preemptive patenting. Kamien and Tauman (1986) and Katz and Shapiro (1985, 1986) re-examine Arrow's theme, but introduce oligopolistic downstream industries with ex-ante symmetric firms and specifically point out the presence of externalities. These authors study models having the following common structure: 1) The inventor announces the licensing procedure (auction, fixed fee, royalty, etc...); 2) Firms decide whether to buy a license (or how much to bid in an auction); 3) Licensed and unlicensed firms compete in the downstream market[12]. A main result is that, from the point of view of the seller, an auction dominates both fixed fees and royalties contracts[13]. In contrast to these authors, Jehiel and Moldovanu

[10] In most auction formats only the winner pays. In the so-called "all-pay auction" also losers pay. Hence their payoff depends on the realized bids, but neither on the final allocation of the good nor on the winner's characteristics.

[11] To put it more abstractly, in Milgrom and Weber's model bidders perceive only two payoff-relevant alternatives ("I win" and "I loose") and the bid is determined by a (conditional) expected difference of payoffs in the two alternatives. In contrast, our bidders may perceive more than two payoff-relevant alternatives ("I win", "The object is not sold", "The object is sold to a competitor", etc...).

[12] Kamien and Tauman look at an inventor that uses fees or royalties, and discuss the relation between drastic innovations and the emergence of monopoly. Katz and Shapiro consider an innovator that sells licenses via first-price sealed-bid auctions with (optimally set) reserve prices and entry fees.

[13] Consumers in the downstream market have opposite preferences.

(1996) allow for ex-ante asymmetries among the downstream competing firms[14];

In all papers mentioned above information is complete: all relevant parameters (e.g., production costs before and after the licensing process), and hence ex-ante and ex-post downstream profits are common knowledge[15]. In his survey, Kamien (1992) emphasizes the need to extend licensing models by introducing some uncertainty about production costs.

This paper is organized as follows: In Section 2 we describe the economic model and the analyzed auction procedures. We focus on the case of two potential buyers bidding for an indivisible object in a second-price, sealed-bid auction where the seller may sometimes keep the object[16]. Besides the revenue enhancing effects of reserve prices and entry fees, the study of these tools allows us to illustrate how related instruments can nevertheless have very different consequences on bidding behavior.

In Section 3 we illustrate in detail two simple settings that fit in our model: the sale of a cost-reducing innovation (negative externalities) and a merger among firms operating in the same industry (positive externalities).

In Section 4 we focus on auctions with reserve prices. We first derive an equilibrium for the case where the reserve price is not binding[17]. In Subsection 4.1 we derive an equilibrium for the case of negative externalities and a binding reserve price. Some types that are, in principle, willing to pay for preemption, choose nevertheless to make irrelevant bids[18]. We next derive the seller's optimal reserve price, and we show that the seller should sometimes announce a reserve price that is strictly lower than her own valuation for the object[19].

[14] they focus on the incentives to participate in an auction for a cost-reducing innovation, and also show how several beliefs about the final allocation might be consistent with equilibrium behavior, leading to multiple equilibria.

[15] Most of the papers considering other settings with externalities assume complete information (see Segal, 1999). Segal also assumes that the principal has the entire bargaining power (i.e, auctions are excluded). These and several other assumptions made by Segal (e.g., only aggregate trade matters) hinder a straightforward comparison between his and our results.

[16] We indicate the changes needed if there are more than two bidders.

[17] In this equilibrium a bidder takes into account both the expected profit if she acquires the object (i.e., her pure valuation net of externalities) and the impact she expects in case her competitor acquires the object.

[18] The lowest relevant bid is strictly higher than the reserve price.

[19] Setting a low reserve price is a way to increase the supply. Another, quite different context in which increasing the supply may be beneficial to the seller is one of common value auctions (see Bulow and Klemperer 1998). The main phenomena in that paper are caused by "winner's curse" effects, whereas here the effects are due to the presence of externalities.

In Subsection 4.2 we look at the case of positive externalities and a binding reserve price. For the case where the externality is non-increasing in the winner's valuation we are able to derive a (rather complex) symmetric equilibrium in pure strategies which involves pooling at the reserve price[20]. For the case where the externality increases in the winner's valuation, we show that equilibria in pure strategies may not exist.

In Section 5 we look at second-price auctions with entry fees. In the case of negative externalities, there is a natural one-to-one correspondence between entry fees and reserve prices. With positive externalities, entry fees and reserve prices do not have the same effect, since there is no analog of pooling with entry fees. For the positive externality case we also show that, no matter what the seller's valuation for the object is, a strictly positive measure of types is excluded from participation in the auction with the optimal entry fee. This result, which sharply contrasts with the usual intuition, stems from the fact that exclusion also mitigates the free-rider effect among buyers[21]. Finally, we consider a simpler class of situations where the (positive) externality term does not depend on the other agent's private information, and we show that, for each relevant entry fee, the seller can find a reserve price that leads to a strictly higher revenue.

In Section 6 we extend our model to $n > 2$ buyers, and illustrate several facts that are not immediately apparent in the $2-$buyer case[22].

Concluding comments are gathered in Section 7. All proofs appear in an Appendix.

## 2. The Model

We consider the following situation: A seller owns an indivisible object. The seller's valuation for the object is $\pi_S$. There are 2 potential buyers. Buyer's $i$ pure valuation for the object (i.e., his profit when he acquires the object) is given by $\pi_i$. Denote by $\pi_{-i}$ the valuation of the other buyer.

If the good is sold to buyer $i$ for a price $p$ , the utilities of the agents are as

[20] An interesting implication of pooling is that sometimes a bidder with lower type may be the winner of the object, thus leading to ex-post inefficiencies (among the set of bidders)

[21] Setting a positive entry fee is a way to reduce the expected number of competitors, which is revenue enhancing when externalities are positive. Another context in which revenue may decline with the number of competitors is one of common value auctions (see Bulow and Klemperer 1998).

[22] For example, we show that the optimal reserve price depends on the number of bidders.

follows: $p$ for the seller; $\pi_i - p$ for buyer $i$; $g_j(\pi_j, \pi_{-j})$ for buyer $j, j \neq i$. We normalize the utilities of the buyers to be zero in case that the seller keeps the object (this case is called the *status-quo*).

The functions $g_k$, $k = 1, 2$ which are common knowledge, are assumed to be differentiable. Note that the first argument of function $g_k$ is always the type of the sufferer $k$, and the second argument is the type of the other agent.

Buyer $i$'s pure valuation[23] is private information, and it is drawn from interval $[\underline{\pi}_i, \bar{\pi}_i]$ according with the density $f_i$, independently of other buyers' valuations. We assume $f_i(\pi) > 0$ for all $\pi \in [\underline{\pi}_i, \bar{\pi}_i]$, and we denote by $F_i$ the cumulative distribution of $f_i$. Moreover, we assume that $\underline{\pi}_i \geq 0$.

Second-price auctions with a reserve price proceed as follows: The seller announces a reserve price $R \geq 0$. The buyers then simultaneously submit bids for the object. Assume without loss of generality that the bids are $b_1 \geq b_2$. If $R > b_1$, the seller keeps the good and no payments are made. If $b_1 \geq R$ , and $b_1 > b_2$ , buyer 1 gets the good and pays to the seller $p = \max(R, b_2)$. The other buyer pays nothing. If $b_1 = b_2 \geq R$ , then each buyer gets the object with probability $\frac{1}{2}$ . The winner pays $p = b_2$, and the other buyer pays nothing.

In second-price auctions with entry fees, the buyers who participate pay an entry fee $E$ at the same time as they submit a bid[24]. The rules of the auction are those of a second-price sealed-bid auction with reserve price $R = 0$. That is, if at least one bidder participates, the good is allocated to the bidder with highest bid. If there is another participating bidder, the winner pays the second highest bid. Otherwise, she gets the good for free. Buyers who choose not to pay the fee (and hence do not bid at the auction) are still affected by the outcome of the auction.

We consider here a *symmetric* setting in the following sense: 1) $\underline{\pi}_1 = \underline{\pi}_2 = \underline{\pi}$ and $\bar{\pi}_1 = \bar{\pi}_2 = \bar{\pi}$ . 2) There exists a function $f : [\underline{\pi}, \bar{\pi}] \rightarrow \Re$ such that $\forall \pi$, $f_1(\pi) = f_2(\pi) = f(\pi)$. 3) There exists a function $g : [\underline{\pi}, \bar{\pi}] \times [\underline{\pi}, \bar{\pi}] \rightarrow \Re$ such that $\forall \pi, \pi'$, $g_1(\pi, \pi') = g_2(\pi, \pi') = g(\pi, \pi')$. Hence, we assume that the externality suffered by agent 1 with type $\pi$ if agent 2 with type $\pi'$ gets the object is the same as the externality suffered by agent 2 with type $\pi$ if agent 1 with type $\pi'$ gets the

[23] The reader may have noticed the following "asymmetry" in our treatment: while we allow externalities to depend on others' characteristics, we assume that a bidder's payoff when he gets the object does not display this dependence. It is, of course, possible to generalize the model allowing for such a feature, but the ensuing phenomena are well-known by Milgrom and Weber's analysis. We prefered the somewhat simpler model in order to focus on the interplay between informational and allocative externalities.

[24] In our context, it would make no difference to assume that the participating bidders can observe who else participates before they submit their bid.

object[25].

Let $D_x g$ denote the derivative of the function $g$ with respect to the first coordinate (i.e., the type of the sufferer), and let $D_y g$ denote the derivative of the function $g$ with respect to the second coordinate (i.e., the type of the causer). Throughout the paper we assume that

$$\forall \pi, \pi' \in [\underline{\pi}, \bar{\pi}], D_x g(\pi, \pi') \leq 1 \tag{2.1}$$

and that

$$\forall \pi \in [\underline{\pi}, \bar{\pi}], D_x g(\pi, \pi) + D_y g(\pi, \pi) < 1 \tag{2.2}$$

The first assumption ensures that the benefit of winning against any competitor, $\pi - g(\pi, \pi')$, is increasing in the winner's type. The second assumption ensures that the function

$$G(\pi) \equiv \pi - g(\pi, \pi)$$

is strictly monotonically increasing on $[\underline{\pi}, \bar{\pi}]$. Both conditions are standard: they are used for the derivation of a separating equilibrium in Proposition 4.1 below (see also the use of their analogs in Milgrom and Weber's paper[26]).

We will speak of the *negative externalities case* if $\forall \pi, \pi' \in [\underline{\pi}, \bar{\pi}],\ g(\pi, \pi') \leq 0$, and of the *positive externalities case* if $\forall \pi, \pi' \in [\underline{\pi}, \bar{\pi}],\ g(\pi, \pi') \geq 0$. We want to emphasize that these definitions make sense only in relation to a given status-quo, which is normalized here to yield zero utility for both bidders.

We focus below on pure-strategy symmetric equilibria of the various auction formats[27].

# 3. Illustrations

## 3.1. Negative externalities: The sale of a patent

Consider 2 firms in a Cournot oligopoly. Firm $i$'s cost of producing quantity $q_i$ of a homogenous product is given by $cq_i$, where $c < 1$. Let $P(Q) = 1 - Q$ be the

[25] This poperty is sometimes called *exchangeability.*

[26] These authors also assume that the gain from winning is increasing in the other bidders' signals, which, translated to our framework, means $D_y g \leq 0$. But this assumption is not necessary for the derivation of an equilibrium in the second-price auction, and we do not impose it here.

[27] The symmetry, risk-neutrality and type-independence assumptions lead to revenue equivalence for symmetric equilibria of sealed-bid formats. For the case of two bidders an English ascending auction is also equivalent to our mechanism, but this does not necessarily hold for more than two agents. (See also Das Varma 1999.)

market-clearing price when the aggregate supplied quantity is $Q = q_1 + q_2 \leq 1$. The Nash equilibrium the profits[28] are given by

$$\pi_1^{sq} = \pi_2^{sq} = \frac{(1-c)^2}{9} \tag{3.1}$$

All parameters in the status-quo are common knowledge.

Consider an inventor that wants to sell a cost-reducing technical innovation protected by a patent. The firm that acquires the patent[29] will be able to produce the good with marginal cost $0 \leq c_i \leq c$. The new, reduced cost $c_i$ is private information to firm $i$ at the time of the patent's sale. After the sale, the new cost structure is revealed to every competitor[30]. If firm $i$ acquires the patent it earns a profit

$$\pi_i^{own} = \frac{(1-2c_i+c)^2}{9} \geq \pi_i^{sq} \tag{3.2}$$

The other firm $j$, $j \neq i$, produces with the old, relatively more costly technology and it earns a profit

$$\pi_j^{ext} = \frac{(1-2c+c_i)^2}{9} \leq \pi_j^{sq} \tag{3.3}$$

We are in the negative externalities case. Relative to the status-quo, we obtain the following:

1. When firm $i$ acquires the patent, its benefit from the innovation is given by

$$\pi_i = \pi_i^{own} - \pi_i^{sq} = \frac{4}{9}(1-c_i)(c-c_i) \tag{3.4}$$

2. The non-acquiring firm $j$ incurs a loss given by

$$\pi_j^{ext} - \pi_j^{sq} = \frac{1}{9}(c_i - c)(2-3c+c_i) \tag{3.5}$$

Note that the loss suffered by the non-acquiring firm is a function of the benefit of the acquiring firm[31] (which is not observable at the time of the auction). By

[28] $sq$ stands for status-quo

[29] We assume that the patent can be sold only to one firm.

[30] To simplify the discussion, we assume below that both firms will produce positive quantities also after one of them acquires the innovation and becomes more efficient.

[31] In this example, the loss of the non-acquiring firm does not depend on its own benefit were it to obtain the patent i.e., it does not depend on $\pi_j$.

equation 3.4 we obtain

$$c_i = \frac{1}{2}\left(1 + c - 3\sqrt{\frac{(1-c)^2}{9} + \pi_i}\right) \tag{3.6}$$

Together with equation 3.5 , this allows us to express the loss of the non-acquiring firm[32] $j$ as:

$$\begin{aligned} \pi_j^{ext} - \pi_j^{sq} &= g_j(\pi_j, \pi_i) = g_j(\pi_i) = \\ &= \frac{c^2}{6} - \frac{c}{3} + \frac{\pi_i}{4} + \frac{c-1}{2}\left(\sqrt{\frac{(1-c)^2}{9} + \pi_i}\right) \end{aligned} \tag{3.7}$$

How much should a firm, say firm 1, bid to acquire the patent ? Firm's 1 valuation is not well-defined since it depends on 1's belief about the likelihood of possible outcomes. To see that, consider two extreme cases: 1) If firm 1 believes that under no circumstance will the patent be sold to firm 2, then its valuation is $\pi_1 = \pi_1^{own} - \pi_1^{sq}$. 2) If firm 1 believes that in case it fails to buy the patent, the seller will surely sell to firm 2, then its valuation is $\pi_1 - \int g_1(\pi_2)d\pi_2 > \pi_1$. Firm 1's belief, on which its bidding strategy will be based, depends both on the nature of the sale mechanism[33], and on the bidding strategy of the other firm. In an equilibrium of a given sale procedure, bidding strategies must be optimal given beliefs, and beliefs must be consistent with the bidding strategies.

### 3.2. Positive externalities: Merger of Competing Firms

Consider 3 firms in a Cournot oligopoly. Firm $i'$s cost of producing quantity $q_i$ of a homogenous product is given by $cq_i + C$ , where $c < 1$. Let $P(Q) = 1 - Q$ be the market-clearing price when the aggregate supplied quantity is $Q = q_1 + q_2 + q_3 \leq 1$. Assuming that the fixed cost $C$ is such that operation is profitable, the Nash-equilibrium profits are given by

$$\pi_1^{sq} = \pi_2^{sq} = \pi_3^{sq} = \frac{(1-c)^2}{16} - C \tag{3.8}$$

[32] Since $\frac{\partial g_j(\pi_j,\pi_i)}{\partial \pi_j} = 0$ and $\frac{\partial g_j(\pi_j,\pi_i)}{\partial \pi i} = \frac{1}{4} + \frac{(c-1)}{4} \cdot \left(\frac{(1-c)^2}{9} + \pi_i\right)^{-\frac{1}{2}} \leq \frac{1}{4}$, we obtain that the function $G(\pi)$ is strictly increasing.

[33] Iin particular, exclusion instruments such as reserve prices and entry fees affect the probability of a sale.

All parameters in the status-quo are common knowledge.

Consider now the situation where firm 3 is up for sale and where firms 1 and 2 bid for it in a takeover battle. The winner $i, i = 1, 2$, can produce with fixed cost $C$ and with marginal cost $0 \leq c_i \leq c$ (imagine some synergy effect) The new, reduced cost $c_i$ is private information to firm $i$ at the time of the contest[34].

If firm $i$ acquires firm 3 it will earn a profit

$$\pi_i^{own} = \frac{(1 - 2c_i + c)^2}{9} - C \tag{3.9}$$

Firm $j$, $j \neq i$, that does not acquire firm 3 will earn a profit

$$\pi_j^{ext} = \frac{(1 - 2c + c_i)^2}{9} - C \tag{3.10}$$

Relatively to the status-quo, we obtain the following:

1. When firm $i$ acquires firm 3, its benefit is given by

$$\pi_i = \pi_i^{own} - \pi_i^{sq} = \frac{(1 - 2c_i + c)^2}{9} - \frac{(1 - c)^2}{16} \geq 0 \tag{3.11}$$

2. The change in profit for the non-acquiring firm $j$ is given by

$$\pi_j^{ext} - \pi_j^{sq} = \frac{(1 - 2c + c_i)^2}{9} - \frac{(1 - c)^2}{16} \tag{3.12}$$

The main thing to note is that the non-acquiring firm obtains a positive benefit if the cost reduction attained by the merged firm is relatively low[35]. Indeed, for $c_i$ such that $c - c_i \leq \frac{1-c}{4}$, we obtain that $\pi_j^{ext} - \pi_j^{sq} \geq 0$. Hence in such a case we obtain a model with positive externalities[36].

[34] As in the previous example we assume that: 1)The new cost structure is revealed after the auction. 2) Both remaining firms produce positive quantities after the takeover.

[35] In that case the loss due to being less efficient in the new environment is fully offset by the gain of having fewer competitors.

[36] Another interesting illustration for the positive externalities case is offered by Katz and Shapiro (1985): Two oligopolists offer incompatible products, and the consumers' utility increases in the size of the group that uses the same product (there are network externalities). If compatibility can be achieved by attaching an "adapter" to one of the products, then one firm will usually bear the cost of the adapter, while the increased compatibility benefits both firms. This creates a free-rider effect, and the incentives to invest in an adapter may be too low.

By equation 3.11 we obtain

$$c_i = \frac{1}{2}\left(1 + c - 3\sqrt{\frac{(1-c)^2}{16} + \pi_i}\right) \tag{3.13}$$

Together with equation 3.12, this allows us to express the benefit of the non-acquiring firm, $g(\pi_j, \pi_i)$ solely as a function[37] of the profit of the acquirer, $\pi_i$.

# 4. Auctions with a Reserve Price

We first analyze the case where the reserve price is not binding[38]. Assuming that buyer 2 bids according to a strategy $\beta(\pi_2)$ which is monotonically increasing and differentiable, buyer's 1 maximization problem given that he has type $\pi_1$ is:

$$\max_b \left( \int_{\underline{\pi}}^{\beta^{-1}(b)} (\pi_1 - \beta(\pi_2)) f(\pi_2)\, d\pi_2 + \int_{\beta^{-1}(b)}^{\bar{\pi}} g(\pi_1, \pi_2) f(\pi_2)\, d\pi_2 \right) \tag{4.1}$$

**Proposition 4.1.** *Assume that $R \leq G(\underline{\pi})$. An equilibrium*[39] *of the second-price auction is given by*

$$b_i(\pi_i) = G(\pi_i). \tag{4.2}$$

It can be shown that when $R \leq G(\underline{\pi})$, the equilibrium displayed in Proposition 4.1 is the unique symmetric equilibrium in pure strategies in which the reserve price is not binding[40].

[37] Note that $D_x g = 0$, and $D_y g = \frac{d(\pi_j^{ext} - \pi_j^{sq})}{dc_i} \frac{dc_i}{d\pi_i} \leq 0$. With Katz and Shapiro's adapter story we can easily generate an example where $D_x g \neq 0$.

[38] i.e., all valuations (including externality effects) lie above the reserve price.

[39] If there are only two bidders and if the reserve price is not binding, there are only two possible physical outcomes (the good ends up in the hands of one of the two bidders). Hence, each bidder perceives only two payoff relevant alternatives and our equilibrium has the same flavor as the one exhibited by Milgrom and Weber in their symmetric model without allocative externalities. Upon winning the object, the marginal type of the other bidder coincides with the bidder's own type.

[40] Standard arguments can be used to show that this is the only pure-strategy symmetric separating equilibrium (in which the reserve price is not binding). The reason why there cannot be pooling in a symmetric equilibrium in pure strategies (in which the reserve price is not binding) is analogous to the argument of Lemma 8.1 below.

We next derive equilibria for auctions with a binding reserve price. The main difficulty is that various types hold different beliefs about the possible final allocation[41]. Since the impact of a loss to the other buyer is different from the impact of the seller keeping the object[42], we obtain an optimal reaction function for each of the two possible beliefs about the final outcome, respectively. The two reaction functions must be combined to form an overall optimal bidding strategy. For the negative externality case, the two intervals of types (each holding another belief) are separate, and we always find an equilibrium in pure strategies. For the positive externality case it is usually impossible to have separate intervals, and an equilibrium in pure strategies (if it exists!) must display a region of pooling.

### 4.1. Negative externalities

Assume that $G(\underline{\pi}) \leq R \leq G(\overline{\pi})$,and consider the type $G^{-1}(R)$ which is given by the unique solution to the equation

$$G(\pi) = R \tag{4.3}$$

Note that $G^{-1}(R) = R + g(G^{-1}(R), G^{-1}(R)) \leq R$ for $g \leq 0$. The interesting part in the determination of equilibrium is the prescription for buyers with valuations in the interval $[G^{-1}(R), R)$. Given a reserve price $R$ , these types are interested in the good purely for preemptive reasons and they are, in principle, willing to pay more than $R$ for preemption. But, given the equilibrium actions of the other bidder, a buyer with valuation in the interval $[G^{-1}(R), R)$ has a chance to get the good only when the other bidder bids less than $R$. In this case the good will not be sold to the competitor, and preemption is therefore not necessary. Hence, bidding zero is ultimately optimal. The lowest relevant bid is $G(R)$ which is strictly above $R$ if $g(R, R) < 0$[43].

**Proposition 4.2.** *Assume that externalities are negative. An equilibrium of the second-price auction with reserve price $R$, $G(\underline{\pi}) \leq R \leq G(\overline{\pi})$ is given by*

$$b_i(\pi_i) = \left\{ \begin{array}{ll} G(\pi_i) & \text{for } \pi_i \geq R \\ 0 & \text{for } \pi_i < R \end{array} \right\} \tag{4.4}$$

[41] A bidder with a relatively high valuation expects the good to be sold for sure, and the effective competition is provided by the other bidder, while for bidders with relatively low valuations the effective competition is provided by the seller's reserve price.

[42] This is normalized here to be zero.

[43] Caillaud and Jehiel (1998) discuss this point for the case of constant negative externalities.

Except for the indeterminacy of bids for types below $R$, the above displayed strategies constitute the unique pure strategy symmetric equilibrium[44].

We now turn to the seller's optimal reserve price policy. The seller's expected revenue is given by:

$$\begin{aligned} U_S(R) &= (F(R))^2\pi_S + 2F(R)(1-F(R))R + \\ & \quad 2\int_R^{\bar{\pi}}(\pi - g(\pi,\pi))(1-F(\pi))f(\pi)d\pi \end{aligned} \tag{4.5}$$

Differentiating this expression with respect to $R$ we obtain:

$$\frac{\partial U_S}{\partial R} = 2F(R)f(R)\left[\pi_S - R + \frac{1-F(R)}{f(R)} + g(R,R)\frac{1-F(R)}{F(R)}\right] \tag{4.6}$$

The thing to note is the extra term involving $g(R,R)$[45]. Assuming an interior maximum, the equation that determines the optimal reserve price is $R_{opt}$ defined by:

$$R_{opt} - \frac{1-F(R_{opt})}{f(R_{opt})} - g(R_{opt},R_{opt}) \cdot \frac{1-F(R_{opt})}{F(R_{opt})} = \pi_S.$$

Since $g(R_{opt},R_{opt}) \leq 0$, it may happen that the seller optimally announces a reserve price which is strictly lower than her own valuation. The intuition is as follows: when the seller sells more often, the buyers are more afraid that the good will fall in the hands of the competitor, and they bid more aggressively. If the seller's valuation is relatively low, the gain of having higher bids fully offsets the loss suffered in cases where the good is sold at a price below valuation.

It is instructive to relate the revenue maximizing and the welfare maximizing levels of $R$[46]. In the negative externality case, we have seen that $R_{opt} < \pi_S$ is possible. It is readily verified that the welfare maximizing reserve price $R_w$ must satisfy $R_w \geq \pi_S$. The expected social welfare as a function of $R$ is given by:

$$W(R) = F(R)^2\pi_S + 2\int_R^{\overline{\pi}}\int_{\underline{\pi}}^{\pi_1}[\pi_1 + g(\pi_2,\pi_1)]dF(\pi_2)dF(\pi_1) \tag{4.7}$$

[44] If $G(\underline{\pi}) > R > \underline{\pi}$, there are two symmetric equilibria in pure strategies: one in which the reserve price is binding and one in which it is not. This is related to the multiplicity of consistent equilibrium beliefs (leading to multiple equilibria) displayed in Jehiel and Moldovanu (1996).

[45] Observe that without externalities the optimal reserve price, $R_{opt}$, satisfies the equation $R_{opt} - \frac{1-F(R_{opt})}{f(R_{opt})} = \pi_S$, and hence $R_{opt} \geq \pi_S$. This confirms the usual economic intuition about the monopolist that restricts supply.

[46] Recall that, without externalities, the monopolist seller sells "too seldom" from an efficiency viewpoint.

We obtain that

$$W'(R) = 2F(R)f(R)(\pi_S - R) - 2\int_R^{\overline{\pi}} g(\pi_2, R)f(R)dF(\pi_2). \tag{4.8}$$

Since $g$ is non-positive, $R_w$ must lie above $\pi_S$ (and hence possibly above $R_{opt}$).

**Example 4.3.** *Let $n = 2$. Each buyer's valuation $\pi_i$ is drawn from the interval $[0, 1]$ with density $f(\pi_i) = 1$. Let the externality be defined by $g(\pi, \pi') \equiv -\frac{1}{2}$ . We obtain that:*

$$\frac{\partial U_S}{\partial R} = (2R)(\pi_S - 2R + 1 - \frac{1-R}{2R}) \tag{4.9}$$

*The optimal reserve price $R_{opt}$, as a function of the seller's valuation $\pi_S$, is as follows:*

$$R_{opt}(\pi_S) = \left\{ \begin{array}{ll} 0, & \text{if } \pi_S \leq 0.8094 \\ \frac{1}{4}\pi_S + \frac{3}{8} + \frac{1}{4}\sqrt{\pi_S^2 + 3\pi_S - \frac{7}{4}}, & \text{if } 0.8094 < \pi_S \leq 1 \\ 1, & \text{if } \pi_S > 1 \end{array} \right\} \tag{4.10}$$

*Note that a seller with a low positive valuation prefers to set a reservation price equal to zero. At the cutoff-value $\pi_S = 0.8094$ the loss of selling below valuation becomes too high, and the optimal reserve price displays a discrete jump (the profit function is continuous though). If , for example, $\pi_S = \frac{1}{4}$ we obtain $R_{opt} = 0$,while the welfare-maximizing reserve price is $R_w = 0.78$. Hence, a revenue-maximizing seller sells "too often".*

### 4.2. Positive Externalities

In this section we study equilibria for the case where the seller imposes a binding reserve price and there are positive externalities.

Assume that $G(\overline{\pi}) \geq R \geq \underline{\pi} \geq 0$ and let again $G^{-1}(R)$ denote the unique solution to the equation $G(\pi) = R$. Note that $G^{-1}(R) = R + g(G^{-1}(R), G^{-1}(R)) \geq R \geq \underline{\pi}$.

We first observe that a pure strategy symmetric separating equilibrium does not exist. To see this, note that in a symmetric equilibrium, buyer $i$ with type $\pi_i$ must bid $G(\pi_i)$ if he bids above $R$[47]. Thus, the only candidates for symmetric separating equilibria are such that buyer $i$ with type $\pi_i$ bids $G(\pi_i)$ for all $\pi_i \geq \pi^*$ and bids zero for $\pi_i < \pi^*$, where $\pi^* \geq G^{-1}(R)$. But, such a strategy profile

[47] Marginally, it is the competition with the other buyer that drives the bidding strategy.

cannot constitute an equilibrium: If bidder 2 bids $G(\pi_2)$ for $\pi_2 \geq \pi^* \geq G^{-1}(R)$, and zero otherwise, then bidder 1 with a type $\pi_1$ slightly below $G^{-1}(R)$ strictly prefers to bid $R$ instead of zero, since this allows her to win the good (thus making an additional strictly positive profit) whenever $\pi_2 < \pi^*$.

Hence, in a symmetric equilibrium there must be some pooling. The intuition is as follows. Bidders with low pure valuation must compete against the seller's reserve price, and they must take into account that no sale is a possible outcome. As a result, such bidders may bid more aggressively than buyers with higher valuations who are sure that a sale will occur[48]. On the other hand, by incentive compatibility arguments, buyers with lower types cannot get the good more often than buyers with higher types. These conflicting forces result in pooling at the bid where the belief about potential outcomes switches, i.e., exactly at the reserve price $R$. If $D_y g \leq 0$[49], an equilibrium can be obtained by a careful construction of the interval of types that pool at $R$ :

1. There is a type $\tilde{\pi}$, $R \leq \tilde{\pi} \leq G^{-1}(R)$,which is indifferent between a bid of zero and a bid equal to $R$, and there is a type $\tilde{\tilde{\pi}}$, $G^{-1}(R) \leq \tilde{\tilde{\pi}} \leq \bar{\pi}$, which is indifferent between any two bids in the interval $[R, G(\tilde{\tilde{\pi}})]$.
2. All types in the interval $[\tilde{\pi}, \tilde{\tilde{\pi}})$ make the **same** bid[50], equal to $R$ (and this bid is strictly preferred to any other bid).
3. All types below $\tilde{\pi}$ bid zero, and, finally, all types $\pi \geq \tilde{\tilde{\pi}}$ bid $G(\pi)$.

The next Lemma characterizes the extremities of the pooling interval $[\tilde{\pi}, \tilde{\tilde{\pi}})$ :

**Lemma 4.4.** *Assume that $D_y g \leq 0$, and that for all $\pi \geq R$, $\bar{\pi} - g(\bar{\pi}, \pi) \geq R$. The system of equations:*

$$\begin{aligned}(u-R)\left(F(u)+F(z)\right)-\int_u^z g(u,\pi)f(\pi)d\pi &= 0\\ (z-R)(F(z)-F(u))-\int_u^z g(z,\pi)f(\pi)d\pi &= 0\end{aligned} \tag{4.11}$$

[48] Note that, with positive externalities, a buyer believing that the good will not be sold (if he himself does not acquire it) is prepared to pay more than than a buyer with the same pure valuation believing that the good will be sold for sure.

[49] Observe that this assumption fits with the positive externality example provided in Section 3.

[50] Haile (1999) illustrates this construction in an auction model with resale opportunities. Since a loser at the auction has a chance to buy the good in the resale market, his model displays positive externalities.

has a solution $(u, z) = (\tilde{\pi}, \tilde{\tilde{\pi}})$ such that $R \leq \tilde{\pi} \leq G^{-1}(R)$ and $G^{-1}(R) \leq \tilde{\tilde{\pi}} \leq \bar{\pi}$.

**Proposition 4.5.** *Assume that $D_y g \leq 0$ and that, for all $\pi \geq R$, $\bar{\pi} - g(\bar{\pi}, \pi) \geq R$. Let $(\tilde{\pi}, \tilde{\tilde{\pi}})$ be a solution of the system 4.11 that satisfies $R \leq \tilde{\pi} \leq G^{-1}(R)$ and $G^{-1}(R) \leq \tilde{\tilde{\pi}} \leq \bar{\pi}$. The strategy profile*

$$b_i(\pi_i) = \left\{ \begin{array}{ll} G(\pi_i) & \text{for } \pi_i \in [\tilde{\tilde{\pi}}, \bar{\pi}] \\ R & \text{for } \pi_i \in [\tilde{\pi}, \tilde{\tilde{\pi}}) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\} \tag{4.12}$$

*constitutes a Nash equilibrium*[51].

An implication of pooling is that even when the good is sold, it is not necessarily sold to the efficient buyer.

Our next result looks at the case where the externality function does not depend at all on the type of the acquirer. In this case the determination of the pooling interval is somewhat simpler, as the upper end of the pooling interval is exactly $G^{-1}(R)$.

If $D_y g(\pi, \pi') = 0$ for all $\pi, \pi' \in [\underline{\pi}, \bar{\pi}]$ we define $h(\pi) \equiv g(\pi, \pi')$. Let $H(u) = (u - R)\left(F(u) + F(G^{-1}(R))\right) - (F(G^{-1}(R)) - F(u))h(u)$ and let $\tilde{\pi}$ be defined[52] by $H(\tilde{\pi}) = 0$.

**Corollary 4.6.** *Assume that $D_y g(\pi, \pi') \equiv 0$ and that $\bar{\pi} - h(\bar{\pi}) \geq R$. The strategy profile*

$$b_i(\pi_i) = \left\{ \begin{array}{ll} G(\pi_i) & \text{for } \pi_i \in [G^{-1}(R), \bar{\pi}] \\ R & \text{for } \pi_i \in [\tilde{\pi}, G^{-1}(R)) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\} \tag{4.13}$$

[51] Assume that for all $\pi \geq R$, and for all $\pi'$ such that $R \leq \pi' \leq \pi$, $g(\pi, \pi') > \pi - R$ (this means, roughly, that the externality in all relevant cases is higher than the gain of acquiring the object). In this case, no matter what $u \geq R$ is, there is no $z \geq u$ such that the second equation in the system 4.11 holds. This implies that the system of equations does not have a solution such that $\tilde{\tilde{\pi}} \in [G^{-1}(R), \bar{\pi}]$. The equilibrium of the auction is then given by $b_i(\pi_i) = R$ for $\pi_i \in [\tilde{\pi}, \bar{\pi})$ and $b_i(\pi_i) = 0$ for $\pi_i \in [\underline{\pi}, \tilde{\pi})$. Type $\tilde{\pi}$ is defined by the first equation in the system 4.11 at $z = \bar{\pi}$. This equation (in the variable $u$) has always a solution $\tilde{\pi}$ on the interval $[R, \bar{\pi}]$.

[52] This is the type that will be indifferent between a bid of $R$ and a bid of zero.

constitutes a Nash equilibrium.

We finally show by way of example that equilibria in pure strategies may fail to exist when the condition $D_y g \leq 0$ is not satisfied[53].

**Proposition 4.7.** *Assume that each buyer's valuation $\pi_i$ is drawn from the interval $[0,1]$ with density $f(\pi_i) = 1$. Let the externality function be given by $g(\pi, \pi') = k\pi'$ where $0 < k < 1$, and let the reserve price $R$ be such that $0 < R < 1 - k$[54]. Then there are no equilibria in pure strategies.*

## 5. Second-Price Auctions with an Entry Fee

Assume now that the seller sells through a second-price auction with an entry fee $E$, $0 < E \leq \overline{\pi}$ (see description in Section 2). Whenever a bidder decides to participate, it is clear that competition is against the other bidder (if any) and not against the seller, who has committed to sell the object. This is the main difference between positive entry fees and positive reserve prices.

The following Proposition characterizes equilibrium behavior in auctions with entry fees (irrespective of the sign of the externalities).

**Proposition 5.1.** *Assume that $G(\underline{\pi}) \geq 0$, and let $\pi^E$ be the unique solution to the equation $E = u \cdot F(u)$. The strategy profile defined by*

$$s_i(\pi_i) = \left\{ \begin{array}{cc} \textit{stay out} & \textit{for } \pi_i \in [\underline{\pi}, \pi^E) \\ \textit{enter, and bid } G(\pi_i) & \textit{for } \pi_i \in [\pi^E, \bar{\pi}] \end{array} \right\} \tag{5.1}$$

*constitutes a Nash equilibrium*[55].

We now compute the seller's revenue in an auction with an entry fee $E$ . Since there is a one-to-one correspondence between $E$ and $\pi^E$, we can write the seller's revenue as a function of $\pi^E$. The revenue is :

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[53] If $D_y g > 0$, we obtain $\tilde{\tilde{\pi}} < G^{-1}(R)$. This means that a type slightly above $\tilde{\tilde{\pi}}$ prefers to bid above $R$, but type $G^{-1}(R)$ never bids above $R$. Hence the *single crossing of incremental returns*, which is a sufficient condition for the existence of pure strategy equilibria, is not satisfied (see Athey, 1999).

[54] The condition $R < 1 - k$ ensures that $G^{-1}(R) = \frac{R}{1-k} < 1 = \overline{\pi}$. If $G^{-1}(R) \geq 1$, then there is an equilibrium of the form $b_i(\pi_i) = \left\{ \begin{array}{cc} R & \text{for } \pi_i \in [\tilde{\pi}, \overline{\pi}) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}) \end{array} \right\}$

[55] This is the unique pure strategy symmetric equilibrium.

$$
\begin{aligned}
U_S(\pi^E) &= (F(\pi^E))^2\pi_S + 2F(\pi^E)(1 - F(\pi^E)E + 2(1 - F(\pi^E)^2E + \\
&\quad 2\int_{\pi^E}^{\bar{\pi}}(\pi - g(\pi,\pi))(1 - F(\pi))f(\pi)d\pi \\
&= (F(\pi^E))^2\pi_S + 2F(\pi^E)(1 - F(\pi^E))\pi^E + \\
&\quad 2\int_{\pi^E}^{\bar{\pi}}(\pi - g(\pi,\pi))(1 - F(\pi)f(\pi)d\pi \qquad (5.2)
\end{aligned}
$$

Differentiating the above expression with respect to $\pi^E$, we obtain:

$$\frac{\partial U_S}{\partial \pi^E} = 2F(\pi^E)f(\pi^E)\left[\pi_S - \pi^E + \frac{1 - F(\pi^E)}{f(\pi^E)} + g(\pi^E,\pi^E)\frac{1 - F(\pi^E)}{F(\pi^E)}\right] \qquad (5.3)$$

For the case of non-positive externalities we obtain that any entry fee policy is revenue equivalent to an appropriately constructed reserve price policy, and vice-versa. Indeed, observe the analogy between the expression above and the respective expression in the reserve price policy (Equation 4.6). In particular, the optimal entry fee is given by $E_{opt} = R_{opt} \cdot F(R_{opt})$.

We now turn to the case of positive externalities, and we first illustrate a rather surprising phenomenon arising in this case.

**Proposition 5.2.** *Assume that $g(\underline{\pi},\underline{\pi}) > 0$. Then, no matter what the seller's valuation is, a positive measure of buyers' types is excluded from participation in the auction with the optimal entry fee.*

The standard economic intuition for the case without externalities is as follows: When the demand parameters (here buyers' valuations) are much larger than the supply parameters (here the seller's valuation), supply restriction (here exclusion) does not make sense since lost sale opportunities cannot be compensated by the higher payments. With positive externalities, exclusion has an additional effect: by selling less often the seller mitigates the free-rider effect among buyers. It is interesting that the free-riding mitigation effect is always stronger than the lost-opportunities effect.

**Example 5.3.** *Each buyer's valuation $\pi_i$ is drawn from the interval $[0,1]$ with density $f(\pi_i) = 1$. Let the externality be $g(\pi_1,\pi_2) \equiv \frac{1}{2}$ . We obtain that:*

$$\frac{\partial U_S}{\partial \pi^E} = (2\pi^E)(\pi_S - 2\pi^E + 1 + \frac{1 - \pi^E}{2\pi^E}) \qquad (5.4)$$

The optimal cut-off type $\pi_{opt}(\pi_S)$ is given by :

$$\pi_{opt}(\pi_S) \quad = \quad \left\{ \begin{array}{ll} \frac{1}{8}+\frac{1}{4}\pi_S+\frac{1}{8}\sqrt{(17+4\pi_S+4(\pi_S)^2\ )}, & \text{if } \pi_S < 1 \\ 1, & \text{if } \pi_S \geq 1 \end{array} \right\}$$

Note that $\pi_{opt}(\pi_S) > 0$ for all $\pi_S$. The optimal entry fee is $E_{opt}(\pi_S) = (\pi_{opt}(\pi_S))^2$. ■

It is also worth noting that, with positive externalities, the welfare maximizing entry fee is always smaller than the revenue maximizing entry fee. Denoting by $W(\pi_E)$ the expected welfare associated with entry fee $E$, we obtain :

$$W(\pi_E) = F(\pi_E)^2\pi_S + 2\int_{\pi_E}^{\overline{\pi}}\int_{\underline{\pi}}^{\pi_1}[\pi_1 + g(\pi_2,\pi_1)]dF(\pi_2)dF(\pi_1) \tag{5.5}$$

$$W'(\pi_E) = 2F(\pi_E)f(\pi_E)(\pi_S - \pi_E) - 2\int_{\pi_E}^{\overline{\pi}} g(\pi_2,\pi_E)f(\pi_E)dF(\pi_2). \tag{5.6}$$

Since $g$ is non-negative, we obtain $W'(\pi_E) < 0$ for $\pi_E > \pi_S$ thus showing that cut-off type $\pi_w$ corresponding to the welfare maximizing entry fee $E_w$ satisfies $\pi_w < \pi_S$. In contrast, expression 5.3 shows that the cut-off type $\pi_{opt}$ corresponding to the revenue maximizing entry fee $E_{opt}$ must satisfy $\pi_{opt} > \pi_S$.

Propositions 5.1 and 4.5 reveal that entry fees and reserve prices are not equivalent in the positive externality case[56]. We prove below that the seller is better-off using a reserve price if the externality term does not depend on the competitor's valuation.

**Proposition 5.4.** *Assume that* $\forall \pi, \pi'$ , $g(\pi,\pi') > 0$, *and that* $G(\underline{\pi}) \geq 0$. *Moreover, assume that for all* $\pi, \pi' \in [\underline{\pi}, \overline{\pi}]$, $D_y g(\pi,\pi') = 0$. *For each auction with an entry fee there is an auction with reserve price that yields a strictly higher revenue for the seller.*

## 6. Extension to $n > 2$ Buyers

We now comment on the extension of our symmetric model to more than two bidders.

[56] The general comparison of the seller's revenue when using an entry fee or a reserve price is quite difficult. It depends on the size of the pooling interval in the auction with a binding reserve price, and in some cases we do not even know whether an equilibrium of the second price auction with reserve price exists - see Proposition 4.7.

Buyers' pure valuations are private information, and they are all independently drawn from the interval $[\underline{\pi}, \bar{\pi}]$ according with the density $f$. We denote by $F$ the distribution of $f$.

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_n)$ . We denote by $\boldsymbol{\pi}_{-ij}$ the vector obtained from $\boldsymbol{\pi}$ by deleting the coordinates $i, j, i \neq j$, and by $\pi_{-ij}^{\max}$ the largest coordinate of $\boldsymbol{\pi}_{-ij}$..

Let $\boldsymbol{\pi}$ be the vector of pure valuations. If the good is sold to buyer $i$ for a price $p$ , the utilities of the agents are as follows: $p$ for the seller; $\pi_i - p$ for buyer $i$; $g_j^i(\pi_j, \pi_i, \boldsymbol{\pi}_{-ji})$ for buyer $j, j \neq i$. The functions $g_j^i$ are common knowledge. A *symmetric* setting is characterized by the existence of a function $g : \Re^N \rightarrow R$, symmetric in its last $n-2$ coordinates, such that if any buyer $i$ with type $\pi_i$ obtains the object, the externality on any buyer $j$, $j \neq i$, with type $\pi_j$ is given by $g(\pi_j, \pi_i, \boldsymbol{\pi}_{-ji})$.

With suitable assumptions that ensure monotonicity[57], an equilibrium in a pure second-price auction is given by:

$$b_i(\pi_i) = \pi_i - \int_{\{\boldsymbol{\pi}_{-ij}/\pi_{-ij}^{\max} \leq \pi_i\}} g(\pi_i, \pi_i, \boldsymbol{\pi}_{-ij}) d\boldsymbol{\pi}_{-ij} \tag{6.1}$$

where $\pi_{-ij}^{\max}$ denoted the maximum of buyers $k$, $k \neq i, j$, types.

The symmetry assumption ensures that the above expression does not depend on the choice of $j$, $j \neq i$, and that all buyers with the same type make the same bid[58] (i.e., the equilibrium is symmetric).

The equilibrium for the negative externality case with a binding reserve price is similar to the one derived for the setting with only two buyers: All types below $R$ bid zero, and types above $R$ bid according to expression 6.1.

A phenomenon which is not apparent for $n = 2$ is the fact that the optimal reserve price does, in general, depend on the number of buyers $n$[59]. For a simple illustration of this dependence, consider the case where for any number $n \geq 2$ of potential buyers, the externality if the good falls in the hands of another is constant, and equal to $\alpha \leq 0$. Then, for each buyer $i$ , the equilibrium bidding

[57] Denote by $D_x g$ the derivative of the externality function with respect to own type, by $D_y g$ the derivative with respect to the type of the winner (when different from oneself) and by $D_z g$ the derivative with respect to the type of another non-acquiring buyer (by symmetry, $D_y g$ and $D_z g$ do not depend on identities). By analogy to Milgrom and Weber's model, sufficient conditions for monotonicity are given by: $1 - D_x g > 0$ ; $-D_y g \geq 0$ ; $-D_z g \geq 0$.

[58] The event that determines the bid is that where one of the other bidders has the same valuation, and all other bidders have a lower valuation.

[59] The optimal reserve price in the symmetric independent private values case without externalities case does not depend on $n$. This is a somewhat surprising, but well known result (see, for example, Myerson (1981)).

strategy is given by $b(\pi) = 0$, for $\pi \in [\underline{\pi}, R)$ and $b(\pi) = \pi - \alpha$, for $\pi \in [R, \bar{\pi}]$. The seller's revenue is given by

$$\begin{aligned} U_S(R) &= F^n(R)\pi_S + nF^{n-1}(R)(1 - F(R))R + \\ & n(n-1)\int_R^{\bar{\pi}} (\pi - \alpha)F^{n-2}(\pi)(1 - F(\pi))f(\pi)d\pi \end{aligned} \tag{6.2}$$

Differentiating this expression with respect to $R$ we obtain:

$$\frac{\partial U_S}{\partial R} = nF^{n-1}(R)f(R)\left[\pi_S - R + \frac{1 - F(R)}{f(R)} - (n-1)\alpha\frac{1 - F(R)}{F(R)}\right] \tag{6.3}$$

The optimal reserve price will depend on $n$ unless the total externality imposed by any buyer, $(n-1)\alpha$, is kept constant as $n$ varies[60].

An equilibrium for the $n$-buyer case with positive externalities (whenever it exists and it is not trivial) displays a region of pooling, as before. The only significant change is the derivation[61] of the critical types $\tilde{\pi}(n)$, $\tilde{\tilde{\pi}}(n)$ . For a simple, example, assume that for any number $n \geq 2$ of potential buyers, the externality is constant, and equal to $\alpha \geq 0$. Assume also that $\underline{\pi} - \alpha \geq 0$. We are then in a similar case to the one covered by Corollary 4.6, and $\tilde{\tilde{\pi}}(n) \equiv R + \alpha$. By keeping $R$ constant, and by maintaining the symmetric tie-breaking rule, one obtains that $\lim_{n\to\infty} \tilde{\pi}(n) = G^{-1}(R) = R + \alpha$[62].

Finally, the equilibrium for the auction with an entry fee is analogous to the $2-$ buyer case. The critical type $\pi^E$ is given by the unique solution to the equation $E = u \cdot (F(u))^{n-1}$ . All types below $\pi^E$ do not enter the auction, and all types above bid according to expression 6.1.

[60] Situations where the suffering decreases if it is shared among many is captured by the old saying: "Misery loves company".

[61] It should be clear from the argument for $n = 2$ that this derivation depends on the number of bidders, not the least through the specification of the tie-breaking rule.

[62] The intuition is as follows: as $n \to \infty$, the probability that the good is eventually sold (even if there is a positive reserve price) goes to 1. Hence, as $n \to \infty$ , a bid of zero becomes attractive for higher valuation types since, in the limit, a payoff of $\alpha$ is assured with probability one. On the other hand, the probability of winning the good with the minimal bid $R$ goes to zero, and this bid is optimal for fewer and fewer types.

# 7. Concluding Remarks

This paper has explored bidding behavior in contexts where there are externalities between bidders, and where these externalities depend on characteristics that may not be observable at the time of the auction. The main driving force is the fact that a buyer's willingness to pay (which determines the bid) depends on several potential scenarios about the final allocation of the good (which is, in turn, determined by the bids at the auction). While studying the effects of standard tools such as reserve prices and entry fees, we have illustrated several important qualitative differences between the cases where externalities are positive or negative.

It is still an open question whether mixed-strategy Nash equilibria exist in auctions with a reserve price when the positive externality increases in the competitor's valuation[63].

Throughout the paper we have abstracted from the possibility that the bids at the auction may serve as signals that influence beliefs in the future interaction. Another relevant extension is obtained by endowing the seller with several objects (licenses, say). But the analysis of bidding behavior in standard multi-object auctions with informational and allocative externalities is likely to be very complex[64]. These subjects will be treated in future work.

# 8. Appendix

## 8.1. Proof of Proposition 4.1

We first assume that buyer 2 bids according to the strategy $\beta(\pi_2)$ which is monotonically increasing and differentiable, and we derive the necessary FOC for buyer 1. Buyer's 1 expected utility given that he has type $\pi_1$, and given that he makes a bid $b$ is :

$$U(\pi_1, b) = \int_{\underline{\pi}}^{\beta^{-1}(b)} (\pi_1 - \beta(\pi_2)) f(\pi_2)\, d\pi_2 + \int_{\beta^{-1}(b)}^{\bar{\pi}} g(\pi_1, \pi_2) f(\pi_2)\, d\pi_2 \tag{8.1}$$

[63]Note that auctions with positive externalities and binding reserve prices do not necessarily satisfy Reny's (1999) "better reply security" sufficient condition. We conjecture that equilibria in mixed strategies do not exist.

[64]See Jehiel and Moldovanu (1999) for a study of difficulties arising in direct, efficient mechanisms.

Differentiating the above expression with respect to $b$ we obtain:

$$\frac{\partial U(\pi_1, b)}{\partial b} = \frac{d\beta^{-1}(b)}{db} f\left(\beta^{-1}(b)\right)\left[\pi_1 - \beta(\beta^{-1}(b)) - g\left(\pi_1, \beta^{-1}(b)\right)\right] \tag{8.2}$$

By symmetry we must have in equilibrium that $\beta^{-1}(b) = \pi_1$. Hence, we obtain:

$$\frac{\partial U(\pi_1, b)}{\partial b} = 0 \Longleftrightarrow b = G(\pi_1) \tag{8.3}$$

We now prove that the strategy $b(\pi_1) = G(\pi_1)$ is optimal for buyer 1 , given that buyer 2 plays the strategy $b(\pi_2) = G(\pi_2)$. Assume that buyer 2 has type $\pi_2$ . When buyer 1 bids above $G(\pi_2)$, he gets the good and his payoff is $\pi_1 - G(\pi_2)$. When he bids below $G(\pi_2)$, buyer 2 gets the good, and buyer 1's payoff is $g(\pi_1, \pi_2)$. By the Mean Value Theorem we have that $\pi_1 - G(\pi_2) - g(\pi_1, \pi_2) = (\pi_1 - \pi_2)(1 - D_x g(\tau, \pi_2))$ , where $\tau$ is between $\pi_1$ and $\pi_2$. By assumption, $1 - D_x g(\tau, \pi_2) \geq 0$. Hence, bidding above $G(\pi_2)$ is optimal if $\pi_1 \geq \pi_2$ , and bidding below $G(\pi_2)$ is optimal if $\pi_1 \leq \pi_2$. By the monotonicity of the function $G(\pi)$ , the bidding function $b(\pi_1) = G(\pi_1)$ satisfies all these optimality requirements for all $\pi_1$. ■

### 8.2. Proof of Proposition 4.2

Assume that buyer 2 bids according to the strategy in the statement of the Proposition. Consider now buyer 1, and assume that $\pi_1 \in [\underline{\pi}, R)$. For such a type, bidding zero (or any other bid below $R$) yields

$$U_1(\pi_1, 0) = \int_R^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \tag{8.4}$$

Bidding $R \leq b \leq G(R)$ yields

$$U_1(\pi_1, b) = \int_{\underline{\pi}}^{R} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_R^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \tag{8.5}$$

Since $\pi_1 \leq R$, the first integral is negative, and bidding zero is preferred to bidding $b$, $R \leq b \leq G(R)$. Finally, bidding $b \geq G(R)$, yields

$$\begin{aligned} U_1(\pi_1, b) &= \int_{\underline{\pi}}^{R} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_{G^{-1}(b)}^{\overline{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \\ &\quad + \int_R^{G^{-1}(b)} (\pi_1 - \pi_2 + g(\pi_2, \pi_2)) f(\pi_2) d\pi_2 \end{aligned} \tag{8.6}$$

Note that

$$\begin{aligned}
& U_1(\pi_1, G(R)) - U_1(\pi_1, b) \\
= & \int_R^{G^{-1}(b)} (\pi_2 - \pi_1 + g(\pi_1, \pi_2) - g(\pi_2, \pi_2)) f(\pi_2) d\pi_2 \\
= & \int_R^{G^{-1}(b)} (\pi_2 - \pi_1)(1 - D_x(\theta, \pi_2)) f(\pi_2) d\pi_2 < 0 \qquad (8.7)
\end{aligned}$$

(The last equality holds for a certain $\theta \in [\pi_1, \pi_2]$ and follows by the Mean Value Theorem). Hence bidding $G(R)$ is preferred to any other bid $b \geq G(R)$. Since we showed above that all bids below $R$ (which are equivalent) are preferred to a bid of $G(R)$, we obtain that a bid of zero is optimal for all $\pi_1 \leq R$.

The proof that bidding $G(\pi_1)$ is optimal for types $\pi_1 \in [R, \bar{\pi}]$ is analogous to the one of Proposition 4.1 and is omitted here. ■

### 8.3. Proof of Lemma 4.4

Fix $u$ such that $R \leq u \leq G^{-1}(R)$, and consider the equation

$$(z - R)(F(z) - F(u)) - \int_u^z g(z, \pi) f(\pi) d\pi = 0 \qquad (8.8)$$

Defining $P(z) = \int_u^z z - R - g(z, \pi) f(\pi) d\pi$, the previous equation becomes $P(z) = 0$. For $z > G^{-1}(R)$ we obtain that

$$P'(z) = \int_u^z (1 - D_x(z, \pi) f(\pi) d\pi + f(z) \cdot (G(z) - R) > 0 \qquad (8.9)$$

By the definition of $G^{-1}(R)$, and by $D_y g \leq 0$, we obtain that $P(G^{-1}(R)) \leq 0$. By the Mean Value Theorem, we obtain that

$$P(\bar{\pi}) = (1 - F(u))(\bar{\pi} - g(\bar{\pi}, \theta) - R) \qquad (8.10)$$

where $u \leq \theta \leq \bar{\pi}$. By the assumption that $\forall \pi \geq R,\ \bar{\pi} - g(\bar{\pi}, \pi) \geq R$, we obtain that $P(\bar{\pi}) \geq 0$. Since the function $P(z)$ is strictly monotonically increasing on the interval $[G^{-1}(R), \bar{\pi}]$ , there exists a unique $z$, $G^{-1}(R) \leq z \leq \bar{\pi}$, such that $P(z) = 0$.

Hence, for each $u, R \leq u \leq G^{-1}(R)$, we have found a unique $z = z(u) \geq G^{-1}(R)$ such that

$$(z(u) - R)(F(z(u)) - F(u)) - \int_u^{z(u)} g(z(u), \pi) f(\pi) d\pi = 0 \qquad (8.11)$$

By the implicit function theorem, the function $z(u)$ is continuous.

Consider now the continuous function

$$H(u) = (u - R)\left(F(u) + F(z(u))\right) - \int_u^{z(u)} g(u,\pi) f(\pi) d\pi. \tag{8.12}$$

Note that

$$\begin{aligned} H(u) &= (u - R)\left(F(u) + F(z(u))\right) - \int_u^{z(u)} g(u,\pi) f(\pi) d\pi \\ &\quad +(u-R)(F(u) - F(z(u))) - (u-R)(F(u) - F(z(u))) \\ &= 2F(z(u))(u-R) + \int_u^{z(u)} (u - R - g(u,\pi)) f(\pi) d\pi \end{aligned} \tag{8.13}$$

We have $H(R) \leq 0$. By the Mean Value Theorem we obtain also that

$$\begin{aligned} H(G^{-1}(R)) &= 2F(z(G^{-1}(R))(G^{-1}(R) - R) \\ &\quad +(F(z(G^{-1}(R))) - F(G^{-1}(R)))(G^{-1}(R) - g(G^{-1}(R),\zeta) - R) \end{aligned}$$

where $G^{-1}(R) \leq \zeta \leq z(G^{-1}(R))$. By the definition of $G^{-1}(R)$ , and by $D_y g \leq 0$, we obtain that $H(G^{-1}(R)) \geq 0$. Hence the equation $H(u) = 0$ has a solution $\widetilde{\pi}$ on the interval $[R, G^{-1}(R)]$ .[65].

The pair $(\widetilde{\pi}, \widetilde{\widetilde{\pi}})$ where $\widetilde{\widetilde{\pi}} = z(\widetilde{\pi})$ is a solution of the system, as required. ■

## 8.4. Proof of Proposition 4.5

Assume that buyer 2 uses the above strategy, and consider a type $\pi_1 \in [\underline{\pi}, G^{-1}(R)]$ of buyer 1. Bidding zero (or any other bid strictly below $R$) yields:

$$U_1(\pi_1, 0) = \int_{\widetilde{\pi}}^{\overline{\pi}} g(\pi_1,\pi_2) f(\pi_2) d\pi_2 \tag{8.14}$$

Bidding $R$ yields:

$$\begin{aligned} U_1(\pi_1, R) &= \int_{\underline{\pi}}^{\widetilde{\pi}} (\pi_1 - R) f(\pi_2) d\pi_2 + \\ &\quad \frac{1}{2}\int_{\widetilde{\pi}}^{\widetilde{\widetilde{\pi}}} (\pi_1 - R + g(\pi_1,\pi_2)) f(\pi_2) d\pi_2 + \end{aligned}$$

[65] The solution need not be unique. It is unique, if, for example, $D_x g \leq 0$. A more general sufficient condition for uniqueness is given by :$\forall z$, the function $\log[g(v,z) \cdot (1 - F(v))]$ is increasing in $v$.

$$
\begin{aligned}
& \int_{\tilde{\tilde{\pi}}}^{\bar{\pi}} g(\pi_1,\pi_2) f(\pi_2) d\pi_2 \\
= \; & \frac{1}{2}(\pi_1 - R)\cdot (F(\tilde{\pi}) + F(\tilde{\tilde{\pi}})) + \\
& \frac{1}{2}\int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} g(\pi_1,\pi_2) f(\pi_2) d\pi_2 + \int_{\tilde{\tilde{\pi}}}^{\bar{\pi}} g(\pi_1,\pi_2) f(\pi_2) d\pi_2
\end{aligned} \tag{8.15}
$$

Type $\tilde{\pi}$ is indifferent between bidding zero and bidding $R$[66], and:

$$
U_1(\tilde{\pi}, R) - U_1(\tilde{\pi}, 0) = 0 \tag{8.16}
$$

Further, we have:

$$
U_1(\pi_1, R) - U_1(\pi_1, 0) = \frac{1}{2}\left[ (\pi_1 - R)(F(\tilde{\pi}) + F(\tilde{\tilde{\pi}})) - \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} g(\pi_1,\pi_2) f(\pi_2) d\pi_2 \right] \tag{8.17}
$$

Note that

$$
\frac{\partial \left( U_1(\pi_1, R) - U_1(\pi_1, 0) \right)}{\partial \pi_1} = F(\tilde{\pi}) + \frac{1}{2}\int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} \left(1 - D_x g(\pi_1,\pi_2)\right) f(\pi_2) d\pi_2 \tag{8.18}
$$

Since $1 - D_x g(\pi_1, \pi_2) \geq 0$, the function $U_1(\pi_1, R) - U_1(\pi_1, 0)$ is increasing in $\pi_1$ . Hence $U_1(\pi_1, R) - U_1(\pi_1, 0) \geq 0$ for all types $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$, and bidding $R$ is better than bidding zero for these types. Similarly, bidding zero is better than bidding $R$ for types $\pi_1 \in [\underline{\pi}, \tilde{\pi}]$. In fact, it easily follows that a bid of zero is optimal for types $\pi_1 \in [\underline{\pi}, \tilde{\pi}]$. Note also that $\tilde{\pi} \geq R$ (since the first equation in system 4.11 does not admit solutions with $u < R$ .)

We now show that a bid of $R$ is optimal for all types $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$. We still need to consider alternative bids $b > R$. There are two cases: Assume first that $G^{-1}(b) \leq \tilde{\tilde{\pi}}$. Then bidding $b > R$ yields :

$$
U_1(\pi_1, b) = \int_{\underline{\pi}}^{\tilde{\tilde{\pi}}} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_{\tilde{\tilde{\pi}}}^{\bar{\pi}} g(\pi_1,\pi_2) f(\pi_2) d\pi_2
$$

Assume next that $G^{-1}(b) > \tilde{\tilde{\pi}}$. Then bidding $b > R$ yields :

[66] This is exactly how this type was constructed.

$$U_1(\pi_1, b) = \int_{\underline{\pi}}^{\tilde{\tilde{\pi}}} (\pi_1 - R) f(\pi_2) d\pi_2 + \int_{\tilde{\tilde{\pi}}}^{G^{-1}(b)} (\pi_1 - G(\pi_2)) f(\pi_2) d\pi_2$$
$$+ \int_{G^{-1}(b)}^{\bar{\pi}} g(\pi_1, \pi_2) f(\pi_2) d\pi_2 \quad (8.19)$$

If $G^{-1}(b) \leq \tilde{\tilde{\pi}}$ we have then that[67]:

$$U_1(\pi_1, R) - U_1(\pi_1, b) = \frac{1}{2} \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} (R - (\pi_1 - g(\pi_1, \pi_2))) f(\pi_2) d\pi_2 \quad (8.20)$$

If $G^{-1}(b) > \tilde{\tilde{\pi}}$ we have then that:

$$U_1(\pi_1, R) - U_1(\pi_1, b) = \frac{1}{2} \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} (R - (\pi_1 - g(\pi_1, \pi_2))) f(\pi_2) d\pi_2 +$$
$$\int_{\tilde{\tilde{\pi}}}^{G^{-1}(b)} (G(\pi_2) - (\pi_1 - g(\pi_1, \pi_2))) f(\pi_2) d\pi_2 (8.21)$$

We need to show that $U_1(\pi_1, R) - U_1(\pi_1, b) \geq 0$ for $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$. Consider first the second integral in equation 8.21. For each $\pi_2 \in [\tilde{\tilde{\pi}}, G^{-1}(b)]$ we obtain by the Mean Value Theorem that $G(\pi_2) - (\pi_1 - g(\pi_1, \pi_2)) = (\pi_2 - \pi_1) \cdot (1 - D_x g(\tau, \pi_2))$, for a certain $\tau \in [\pi_1, \pi_2]$. Since $D_x g(\tau, \pi_2) \leq 1$, each term in the summation is non-negative, and therefore the integral is non-negative.

Consider now the first integral in equation 8.21 (which is also the only expression appearing in equation 8.20), and let $K(\pi_1) = \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} (R - (\pi_1 - g(\pi_1, \pi_2))) f(\pi_2) d\pi_2$.

Observe that $K(\tilde{\tilde{\pi}}) = 0$ [68]. This shows, in particular, that the type $\tilde{\tilde{\pi}}$ is indifferent between bidding $R$, and bidding any bid $b \in (R, G(\tilde{\tilde{\pi}})]$. Note also that $K'(\pi_1) = \int_{\tilde{\pi}}^{\tilde{\tilde{\pi}}} (-1 + D_x g(\pi_1, \pi_2)) f(\pi_2) d\pi_2 \leq 0$, and hence that $K(\pi_1) \geq 0$ for $\pi_1 \in [\tilde{\pi}, \tilde{\tilde{\pi}}]$. This completes the proof that a bid of $R$ is optimal for all types in the interval $[\tilde{\pi}, \tilde{\tilde{\pi}}]$.

The proof that bidding $G(\pi_1)$ is optimal for types $\pi_1 \in [\tilde{\tilde{\pi}}, \bar{\pi}]$ is analogous to the one in Proposition 4.1, and is omitted here. ■

[67] Note that we have used the assumption that $\tilde{\tilde{\pi}} \geq G^{-1}(R)$ to derive the two expressions above.

[68] This is how $\tilde{\tilde{\pi}}$ was constructed .

## 8.5. Proof of Corollary 4.6

Let $h(\pi) \equiv g(\pi, \pi')$. The function

$$H(u) = (u - R)\left(F(u) + F(G^{-1}(R))\right) - (F(G^{-1}(R)) - F(u))h(u) \tag{8.22}$$

is continuous. Since $G^{-1}(R) \geq R$ and $h(u) \geq 0$, it holds that:

$$H(R) \;=\; -(F(G^{-1}(R)) - F(R))h(R) \leq 0; \tag{8.23}$$

$$H(G^{-1}(R)) = 2\left(G^{-1}(R) - R\right) F(G^{-1}(R)) \geq 0. \tag{8.24}$$

Hence there exists $\widetilde{\pi} \in [R, G^{-1}(R)]$ such that $H(\widetilde{\pi}) = 0$. The system of equations 4.11 becomes now

$$\begin{aligned} (u - R)\left(F(u) + F(z)\right) - (F(z) - F(u))h(u) &= 0 \\ (z - R)(F(z) - F(u) - (F(z) - F(u))h(z) &= 0 \end{aligned} \tag{8.25}$$

We now show that the pair $(u, z) = (\widetilde{\pi}, G^{-1}(R))$ satisfies this system of equations. The first equality in the system holds for this pair by the definition of $\widetilde{\pi}$. The second equality holds since $G^{-1}(R) - R - h(G^{-1}(R)) = G^{-1}(R) - R - g(G^{-1}(R), G^{-1}(R)) = 0$. The claim follows then by the proof of Proposition 4.5. ■

## 8.6. Proof of Proposition 4.7

Standard incentive-compatibility arguments imply that the equilibrium probability of winning must be weakly increasing in type. It is then enough to show that there is no pure strategy equilibrium in which the bidding strategies are weakly increasing in type[69].

[69] The weak monotonicity argument yields the following: Assume that there is an equilibrium in which bidder $i$'s strategy displays a region $[\pi', \pi'']$ where bids are strictly decreasing, and let $[b'', b']$ be the corresponding bid range. Then there cannot be equilibrium bids of bidder $j$ that lie in $[b'', b']$. Replacing the decreasing part of $i's$ strategy by a constant bid (in the range $[b'', b']$) yields another equilibrium without the decreasing region. For non-existence of a pure strategy equilibrium it is hence enough to show non-existence of a pure strategy equilibrium where strategies are weakly increasing.

**Lemma 8.1.** *Let $b_i(\pi_i)$ be such that $b_i(\pi_i) = b^* > R$ for all $\pi_i$ in an interval $[\pi_a, \pi_b]$. Assume that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that for $j \neq i$,*

$$\Pr(b_j(\pi_j) \in (b^* - \varepsilon, b^*]) > \delta \tag{8.26}$$

*Then $b_i(\pi_i)$ cannot be part of an equilibrium strategy profile.*

**Proof.** Consider first a symmetric equilibrium profile $(b_1, b_2)$ such that the function $b = b_1 = b_2$ is constant on an interval as above[70]. By the definition of an equilibrium, type $\pi_a$ of bidder 1 prefers bidding $b^*$ rather than $b^* - \varepsilon$, where $\varepsilon > 0$. This yields

$$(\pi_a - b^*)(\pi_b - \pi_a) \geq \int_{\pi_a}^{\pi_b} k\pi_2 d\pi_2 \tag{8.27}$$

Analogously, type $\pi_b$ prefers bidding $b^*$ rather than $b^* + \varepsilon$, which yields:

$$(\pi_b - b^*)(\pi_b - \pi_a) \leq \int_{\pi_a}^{\pi_b} k\pi_2 d\pi_2 \tag{8.28}$$

The last two equations yield $\pi_a \geq \pi_b$, which is a contradiction[71].

Consider now the case of an asymmetric bidding profile, and assume that bidder's 1 strategy exhibits pooling at a level $b^*$ where $b^*$ is in the range of $b_2$ (see assumption 8.26). If bidder's 2 strategy also displays pooling at $b^*$, then we conclude by the same argument as in the symmetric case. Otherwise, the optimal bid for any $\pi_1 \in [\pi_a, \pi_b]$ is $\pi_1 - kb_2^{-1}(b^*)$, which cannot be a constant. ■

Consider now the following class of strategies:

$$b_i(\pi_i) = \left\{ \begin{array}{cc} L_i(\pi_i) & \text{for } \pi_i \in [\tilde{\tilde{\pi}}_i, \bar{\pi}] \\ R & \text{for } \pi_i \in [\tilde{\pi}_i, \tilde{\tilde{\pi}}_i) \\ 0 & \text{for } \pi_i \in [\underline{\pi}, \tilde{\pi}_i) \end{array} \right\} \tag{8.29}$$

where $L_i$ is a strictly increasing function. Lemma 8.1 shows that, in equilibrium, effective pooling can only take place at the reserve price. Hence, if a pure-strategy equilibrium exists, there exists a pair of strategies in the above class that form an equilibrium. We now show that no such pair exists. Consider first the case of a symmetric equilibrium where strategies have the above form. The system of equations 4.11 reads now

[70] In this case assumption 8.26 is automatically satisfied.

[71] Note that this argument does not work for $b^* = R$, since then a bid $b^* - \varepsilon$ has other consequences.

$$
\begin{aligned}
(u-R)(u+z)-\int_u^z k\pi_2 d\pi_2 &= 0 \\
(z-R)(z-u)-\int_u^z k\pi_2 d\pi_2 &= 0
\end{aligned} \tag{8.30}
$$

The only relevant solution of the system is given by: $u = \tilde{\pi} = \frac{2R}{2-k^2}$, $z = \tilde{\tilde{\pi}} = \frac{2R(k+1)}{2-k^2}$. As in the proof of Proposition 4.5, it is necessary for an equilibrium that $\tilde{\tilde{\pi}} \geq G^{-1}(R) = \frac{R}{1-k}$. However, $\frac{2R(k+1)}{2-k^2} < \frac{R}{1-k}$ for all $k > 0$. This concludes the proof that no symmetric equilibrium in pure strategies exists.

Consider now an asymmetric equilibrium having the form given in 8.29. Type $\tilde{\tilde{\pi}}_1$ must be indifferent between bidding $R$ and bidding $R+\varepsilon$, yielding:

$$
(\tilde{\tilde{\pi}}_1 - R)(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1) = \int_{\tilde{\pi}_2}^{\tilde{\tilde{\pi}}_2} k\pi_2 d\pi_2 = \frac{1}{2}k(\tilde{\tilde{\pi}}_2 - \tilde{\pi}_2)^2 \tag{8.31}
$$

For type $\tilde{\tilde{\pi}}_2$ we obtain analogously:

$$
(\tilde{\tilde{\pi}}_2 - R)(\tilde{\tilde{\pi}}_2 - \tilde{\pi}_2) = \int_{\tilde{\pi}_1}^{\tilde{\tilde{\pi}}_1} k\pi_1 d\pi_1 = \frac{1}{2}k(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1)^2 \tag{8.32}
$$

Combining equations 8.31 and 8.32 we obtain:

$$
\tilde{\tilde{\pi}}_1 - R = \frac{k^3}{8}\frac{(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1)^3}{(\tilde{\tilde{\pi}}_2 - R)^2} \tag{8.33}
$$

Assume without loss of generality that $\tilde{\tilde{\pi}}_1 < \tilde{\tilde{\pi}}_2$ (if these are equal than it immediately follows that $\tilde{\pi}_1 = \tilde{\pi}_2$, and we are in the case of a symmetric equilibrium candidate). Equation 8.33 yields then

$$
\tilde{\tilde{\pi}}_1 - R < \frac{k}{2}(\tilde{\tilde{\pi}}_1 - \tilde{\pi}_1) \tag{8.34}
$$

Since $k < 1$, and since $\tilde{\pi}_1 \geq R$ (see the proof of Proposition 4.5) we obtain a contradiction. ■

## 8.7. Proof of Proposition 5.1

All types that decide to pay the fee face a second-price auction with a zero reserve price. The fact that the bid $G(\pi_i)$ is optimal for a type $\pi_i$ that enters the auction follows in the same manner as in Proposition 4.1. It remains to show that the respective entry/non-entry decisions are optimal. Consider the type $\pi^E$ of buyer 1, and assume that buyer 2 plays according to strategy $b_2$ . By staying out, the payoff of type $\pi^E$ is given by

$$\int_{\pi^E}^{\bar{\pi}} g(\pi^E, \pi_2) f(\pi_2) d\pi_2 \tag{8.35}$$

By entering and bidding $G(\pi^E)$ , his payoff is given by

$$-E + F(\pi^E) \cdot \pi^E + \int_{\pi^E}^{\bar{\pi}} g(\pi^E, \pi_2) f(\pi_2) d\pi_2 = \int_{\pi^E}^{\bar{\pi}} g(\pi^E, \pi_2) f(\pi_2) d\pi_2 \tag{8.36}$$

Hence type $\pi^E$ is indifferent between entering and staying out[72]. It is then straightforward to show that all types $\pi_1 > \pi^E$ strictly prefer to enter the auction. ■

## 8.8. Proof of Proposition 5.2

By equation 5.3, we obtain

$$\frac{\partial U_S}{\partial \pi^E}\Big|_{\pi^E=\underline{\pi}} = 2f(\underline{\pi})g(\underline{\pi}, \underline{\pi}) > 0$$

Hence, any maximizer of $U_S$ , $\pi_{opt}(\pi_S)$, must exceed $\underline{\pi}$. If the seller uses the entry fee $E_{opt}(\pi_S) = \pi_{opt}(\pi_S) \cdot F(\pi_{opt}(\pi_S))$, all types in the interval $[\underline{\pi}, \pi_{opt}(\pi_S))$ do not pay the fee and stay out. ■

## 8.9. Proof of Proposition 5.4

Consider an auction with entry fee $E$, and let $\pi^E$ be the unique solution to the equation $E = uF(u)$. We assume below that $\pi^E \leq \bar{\pi}$. Otherwise, the claim of the proposition is immediate. We now construct[73] an auction with a reserve price $R^E$ where the set of active types (i.e., types that bid at least the reserve price) is exactly $[\pi^E, \bar{\pi}]$ . Define

$$\begin{aligned} H(R) &= \left(\pi^E - R\right)\left(F(\pi^E) + F(G^{-1}(R)\right) \\ &\quad - \left(F(G^{-1}(R) - F(\pi^E)\right) h(\pi^E) \end{aligned} \tag{8.37}$$

[72] The equality in the expression above follows by the definition of $\pi^E$.

[73] The construction is the converse of the one used to determine the type $\tilde{\pi}$ in Corollary 4.6.

Note that $H(R)$ is well-defined and continuous in the interval $[G(\pi^E), \pi^E]$ . We obtain that[74]:

$$H(G(\pi^E)) = 2F(\pi^E)h(\pi^E) > 0; \tag{8.38}$$

$$H(\pi^E) = -\left(F(G^{-1}(\pi^E)) - F(\pi^E)\right) h(\pi^E) < 0 \tag{8.39}$$

Hence, the equation $H(R) = 0$ has a solution $R^E$ in the interval $[G(\pi^E), \pi^E]$.

By the construction of $R^E$ , and by the proof of Corollary 4.6, equilibrium behavior in an auction with reserve price $R^E$ is given by

$$b_i(\pi_i) = \left\{ \begin{array}{ll} G(\pi_i) & \text{for } \pi_i \in \left[G^{-1}(R^E), \bar{\pi}\right] \\ R^E & \text{for } \pi_i \in [\pi^E, G^{-1}(R^E)) \\ 0 & \text{for } \pi \in [\underline{\pi}, \pi^E) \end{array} \right\} \tag{8.40}$$

If all types $\pi_i$ in the interval $[\pi^E, G^{-1}(R^E))$ were to bid $G(\pi_i)$ in the auction with reserve price $R^E$, then this auction would be revenue equivalent to the auction with entry fee $E = \pi^E F(\pi^E)$. But, equilibrium behavior in the auction with reserve price $R^E$ requires that all types $\pi_i \in [\pi^E, G^{-1}(R^E))$ bid instead $R^E$. Since $G(\pi_i) < R^E$ for $\pi_i < G^{-1}(R^E)$ , the seller's revenue in the auction with reserve price $R^E$ is strictly higher than the revenue in the auction with entry fee $E$ (although both auctions induce the same interval of active types. ■

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[74] For the second inequality, note that $G^{-1}(\pi^E) > \pi^E$.

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