Multidimensional Screening, Affiliation, and Full Separation

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Multidimensional Screening, Affiliation, and Full Separation*

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Abstract

We solve a class of two-dimensional screening problems in which one dimension has the standard features, while the other dimension is impossible to exaggerate and enters the agent's utility only through the message but not the true type. Natural applications are procurement and regulation where the producer's ability to produce quality and his costs of producing quantity are both unknown; or selling to a budget constrained buyer. We show that under these assumptions, the orthogonal incentive constraints are necessary and sufficient for the full set of incentive constraints. Provided that types are affiliated and all the conditional distributions of types have monotonic inverse hazard rates, the solution is fully separating in both dimensions.

1 Introduction

The optimal screening of agents has had many fruitful applications including optimal taxation, non linear pricing, public utility regulation and procurement policies. For the most part these studies only consider cases where the agents differ in one unknown characteristic. This restriction is primarily for technical and not for economic reasons.

Formal analysis of multidimensional screening problems is substantially different from the analysis of one-dimensional problems in part because *bunch-*

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ing is a common feature of the optimal solution.[1] Because of these technical difficulties, most of the research using models of adverse selection are one dimensional. There are exceptions. Explicit solution have been found for some multi-dimensional problems; for example, Laffont, Maskin, and Rochet [1987], Lewis and Sappington [1988], McAfee and MacMillan [1988], Matthews and Moore [1987], and Jehiel et al [1999]. In addition there are two useful surveys of these multidimensional problems—Armstrong and Rochet [1999] and Rochet and Stole [2003]. The latter propose two different procedures for reducing the dimensionality of a problem—aggregation and separability. The aggregation procedure consists of finding an aggregator for the unknown characteristics which may of course depend on other parameters of the problem. They refer to separability for the case when the solution to the problem depends upon a particular distribution of types such as in Wilson [1993] and Armstrong [1996]. In this note we propose an alternative, but not unrelated, procedure for reducing the dimensionality of such a problem. We present two assumptions that may hold in many types of models and that imply that the optimum can be found by analyzing a series of one-dimensional problems.

More specifically, we assume that one of the unobserved characteristics is such that individuals can only imitate in one direction and that the two unknown characteristics are affiliated.[2] When these two conditions are met, we show that solution to each of the one-dimensional problems, conditional on the values taken by the characteristic that has an upper bound for each individual, actually yields the solution for the overall problem provided that the two types are affiliated.[3] In particular we show that that the large number of incentive compatibility constraints in this two-dimensional problem can be reduced to two one-dimensional incentive compatibility conditions—one in each direction.

In Section 2 we present a simple principal-agent procurement model in some detail. The principal desires a variable quantity of a good of unknown quality. The agents supply a quantity of the good of a certain quality with unknown costs. It is assumed that each agent can produce a maximum quality and can only imitate lower qualities. The principal is assumed to want each agent to

[1] Bunching refers to the fact that different types may be treated identically in the optimum. The most general results to date have been obtained by Armstrong [1996] and Rochet and Chone [1998].

[2] Two random variables with density $f(\cdot,\cdot)$ are affiliated if for $\theta \geq \theta'$ and $\eta \geq \eta'$

$$f(\theta,\eta) f(\theta',\eta') \geq f(\theta',\eta) f(\theta,\eta')$$

[3] To the best of our knowledge, this was first proposed and used by Beaudry, Blackorby, and Szalay [2006] to solve an optimal income tax problem.

produce the highest quality possible. In this setting we show that the solution to the one-dimensional unknown cost problem—conditional on the quality—is, in fact, the solution to the two-dimensional problem provided that the two unknown characteristics are affiliated.

In Section 3 we set up a reasonably standard regulation problem as in Baron and Myerson [1982], Lewis and Sappington [1988], and Armstrong [1999] allowing for two dimensions of asymmetric information. The regulator maximises a weighted sum of consumers' plus producers' surplus where again the quality and the costs are unknown. Given some restrictions on the surplus functions we show that the solutions to all the one dimensional problems—conditional on quality—remain the solutions to the overall problem provided that the unknown characteristics are affiliated.

In Section 4 we take the model of Che and Gale [2000][4] of selling to an budget-constrained buyer. The buyer has private information about his valuation of the good and about how much money he can spend on this good. Thus the selling procedure may be altered because the buyer cannot spend more money than he has. Che and Gale solve this problem for two cases, when the buyer must post a bond and is therefore unable to mimic more wealthy types, and when the buyer does not have to post a bond. We show that the first case is amenable to the technique proposed in Section 2 and solve this problem by assuming that the valuation of the commodity and the negative of the wealth are affiliated.

2 The procurement model

A principal wishes to contract with an agent to buy a good where x is the quantity of the good the principal obtains, q is the quality of the good, and t is the transfer payment to the agent. . The net utility to the principal is

$$V(x,q)-t.$$

The agent's utility from delivering the good in quantity x and quality q is

$$t-\mathcal{C}(x,q,\theta,\eta)$$

where θ and η are parameters that shift the agent's cost of production. More specifically, we assume that η defines the upper bound on the quality q the agent

[4] See also Che and Gale [1998].

is capable of producing, in the sense that

$$\mathcal{C}(x,q,\theta,\eta)=\begin{cases} C(x,q,\theta) & \text{for } q\leq\eta \\ \infty & \text{for } q>\eta. \end{cases}$$

$C(x,q,\theta)$ is the cost function "on the relevant range of qualities".

We assume that $V(x,q)$ satisfies $V(0,q)=0$ for all q; $V_x(x,q)>0$ and $V_{xx}(x,q)<0$ for all x and q; $\lim_{x\to 0}V_x(x,q)=\infty$ for all q; and $V_q(x,q)>0$ for all $x>0$. $C(x,q,\theta)$ satisfies $C(0,q,\theta)=0$ for all q,θ; $C_x(x,q,\theta)>0$, $C_\theta(x,q,\theta)>0$, and $C_q(x,q,\theta)\geq 0$ for x,q,θ such that $x>0$; and $C_{x\theta}(x,q,\theta)>0$ and $C_{xx}(x,q,\theta)\geq 0$ for all x,q,θ.

These conditions are standard except perhaps for the one which implies that the principal has a taste for higher quality commodities. In addition we need one assumption that is imposed jointly on the value function of the principal and the cost function of the agent: V and C satisfy $V_{xq}(x,q)\geq C_{xq}(x,q,\theta)$. As will be seen in Proposition 1, this guarantees that when there is full information the principal wants to buy the highest quality that is available from the agent.

While the agent knows the parameters θ, and η, the principal knows only their joint distribution. θ,η are distributed on a rectangle $\left[\underline{\theta},\overline{\theta}\right]\times\left[\underline{\eta},\overline{\eta}\right]$ where $\underline{\theta},\underline{\eta}>0$. We denote $f(\theta,\eta)$ the joint density of the distribution and assume it has full support, so that the conditional densities of θ conditional on η, $f(\theta\,|\eta)$, have full support as well. The parameters θ and η are not observable, either ex ante or ex post. The quality of the good, q, and the quantity of the good, x are observable through the utility the principal derives from consuming the good, $V(x,q)$.

Before we analyze the contacting problem in detail, we discuss the benchmark case where the principal has complete information.

2.1 The Full Information Benchmark

The principal's problem is to choose schedules $x(\theta,\eta)$, $t(\theta,\eta)$ and $q(\theta,\eta)$ such that $q(\theta,\eta)\leq\eta$ to maximize her surplus subject to the constraint that the agent is willing to participate. Clearly, the agent's participation constraint,

$$t(\theta,\eta)-C(x(\theta,\eta),q(\theta,\eta),\theta)\geq 0,$$

must be binding for each type, because the principal's net utility is decreasing in t. Imposing this condition, we can write the principal's problem under complete information as follows:

$$\max_{x(\cdot,\cdot),q(\cdot,\cdot)} \int_{\underline{\eta}}^{\overline{\eta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(V\left(x\left(\theta,\eta\right),q\left(\theta,\eta\right)\right) - C\left(x\left(\theta,\eta\right),q\left(\theta,\eta\right),\theta\right)\right) f\left(\theta,\eta\right) d\theta d\eta$$

$$\text{subject to} \quad q\left(\theta,\eta\right) \le \eta.$$

Given our assumptions, the integrand is concave in x, so the optimal quantity schedule, $x^*\left(\theta,\eta\right)$, must satisfy the condition

$$V_x\left(x^*\left(\theta,\eta\right),q\left(\theta,\eta\right)\right) = C_x\left(x^*\left(\theta,\eta\right),q\left(\theta,\eta\right),\theta\right)$$

The optimal quality choice must satisfy either

$$V_q\left(x^*\left(\theta,\eta\right),q^*\left(\theta,\eta\right)\right) = C_q\left(x^*\left(\theta,\eta\right),q^*\left(\theta,\eta\right),\theta\right)$$

or

$$q^*\left(\theta,\eta\right) = \eta \text{ and } V_q\left(x^*\left(\theta,\eta\right),\eta\right) - C_q\left(x^*\left(\theta,\eta\right),\eta,\theta\right) \ge 0$$

In the former case the problem admits an interior solution, whereas, in the latter case it is optimal to produce the highest quality level. We focus on the second option because the parameter η affects the solution only when the constraint is binding. When the constraint is not binding this becomes a one-dimenional problem for which methods of solution are well known. There are simple sufficient conditions that ensure that the solution for q is indeed on the boundry. When the agent's cost is independent of q, then the solution is clearly to produce the highest quality, since the principal benefits from higher quality. Since this case is somewhat trivial, we focus on another set of sufficient conditions.

Proposition 1 *Given our regularity conditions, including the assummption that* $V_{xq}\left(x,q\right) \ge C_{xq}\left(x,q,\theta\right)$,

$$q^*\left(\theta,\eta\right) = \eta \text{ for all } \theta,\eta$$

Proof. Observe that

$$V_q\left(x,q\right) - C_q\left(x,q,\theta\right) = \int_0^x \left(V_{\xi q}\left(\xi,q\right) - C_{\xi q}\left(\xi,q,\theta\right)\right) d\xi + \left(V_q\left(0,q\right) - C_q\left(0,q,\theta\right)\right)$$

Using $V(0,q) = C(0,q,\theta) = 0$ for all q, θ, we have

$$V_q(0,q) = C_q(0,q,\theta) = 0 \text{ for all } q, \theta.$$

Thus, we can write

$$V_q(x,q) - C_q(x,q,\theta) = \int_0^x (V_{\xi q}(\xi,q) - C_{\xi q}(\xi,q,\theta))\, d\xi.$$

By assumption the integrand is non-negative pointwise, which establishes the claim. ■

Higher quality is always desirable because the marginal utility of consuming x increases faster (or at the same speed) in q than does the marginal cost of production. Hence, with the full information the principal always chooses the maximum quality the agent can produce. We now address the principal's problem when θ and η are not observable to him.

2.2 The Principal's Contracting Problem

We analyze the principal's problem as a message game, where the agent is asked to announce a type $\left(\hat{\theta}, \hat{\eta}\right)$ and is given incentives to do so truthfully. Formally, the principal solves the problem

$$\max_{x(\cdot,\cdot),t(\cdot,\cdot),q(\cdot,\cdot)} \int_{\underline{\eta}}^{\overline{\eta}} \int_{\underline{\theta}}^{\overline{\theta}} (V(x(\theta,\eta), q(\theta,\eta)) - t(\theta,\eta)) f(\theta,\eta)\, d\theta d\eta \tag{1}$$

subject to, for all θ, η :

$$t(\theta,\eta) - C(x(\theta,\eta), q(\theta,\eta), \theta) \geq t\left(\hat{\theta},\hat{\eta}\right) - C\left(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right), \theta\right) \text{ for all } q\left(\hat{\theta},\hat{\eta}\right) \leq \eta, \tag{2}$$

$$t(\theta,\eta) - C(x(\theta,\eta), q(\theta,\eta), \theta) \geq 0, \quad \text{and} \tag{3}$$

$$q(\theta,\eta) \leq \eta. \tag{4}$$

While the participation constraint (3), and the feasibility constraint (4) are straightforward to understand, the incentive constraint (2) deserves some explanations. Since the principal cannot verify the agent's true cost of production, the agent can announce any $\hat{\theta}$. On the other hand, the principal can know the quality level of the good the agent delivers by her consumption of it. The agent can only send messages $\left(\hat{\theta}, \hat{\eta}\right)$ such that he is able to deliver the quality level specified in the contract for these messages, $q\left(\hat{\theta},\hat{\eta}\right) \leq \eta$.

A first step toward solving this problem is to show that at the second best solution, the highest quality is produced for each type. This is true only under some additional conditions. In particular, we have the following result:

Proposition 2 *Suppose that the cost function can be written as*

$$C\left(x,q,\theta\right) = c\left(x,\theta\right) + k\left(x,q\right) \tag{5}$$

Then $q^*\left(\theta,\eta\right) = \eta$ *for all* θ,η.

Proof. Take any incentive compatible allocation given by the triple of schedules $x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)$, and $t\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$. Suppose for some $\hat{\theta},\hat{\eta}$, we have $q\left(\hat{\theta},\hat{\eta}\right) < \hat{\eta}$. Let $\tilde{\Theta}$ denote the set of $\left(\hat{\theta},\hat{\eta}\right)$ such that $q\left(\hat{\theta},\hat{\eta}\right) < \hat{\eta}$. Then, we can change the allocation to the new triple of schedules $\tilde{x}\left(\hat{\theta},\hat{\eta}\right), \tilde{q}\left(\hat{\theta},\hat{\eta}\right)$, and $\tilde{t}\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$ as follows. We set $\tilde{x}\left(\hat{\theta},\hat{\eta}\right) = x\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$ and set $\tilde{q}\left(\hat{\theta},\hat{\eta}\right) = \hat{\eta}$ for all $\left(\hat{\theta},\hat{\eta}\right)$. For all $\left(\hat{\theta},\hat{\eta}\right) \in \tilde{\Theta}$ we adjust the transfers from the initial transfers $t\left(\hat{\theta},\hat{\eta}\right)$, to the new transfers

$$\tilde{t}\left(\hat{\theta},\hat{\eta}\right) = t\left(\hat{\theta},\hat{\eta}\right) + C\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\eta},\hat{\theta}\right) - C\left(x\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right).$$

We first show that the new allocation with the new transfers is incentive compatible. Then, we show that the surplus to the principal has increased under the new allocation.

Incentive compatibility of the new system is given if and only if

$$t\left(\theta,\eta\right) + C\left(x\left(\theta,\eta\right),\eta,\theta\right) - C\left(x\left(\theta,\eta\right),q\left(\theta,\eta\right),\theta\right) - C\left(x\left(\theta,\eta\right),\eta,\theta\right) \tag{6}$$

$$\geq \quad t\left(\hat{\theta},\hat{\eta}\right) + C\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\eta},\hat{\theta}\right) - C\left(x\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right) - C\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\eta},\theta\right) \tag{7}$$

From incentive compatibility of $x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)$, and $t\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$ we have

$$t(\theta,\eta) - C\Bigg(x(\theta,\eta), q(\theta,\eta), \theta\Bigg) \geq t\left(\hat{\theta},\hat{\eta}\right) - C\Bigg(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right), \theta\Bigg) \qquad (8)$$

Adding and substracting $C\Bigg(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right), \theta\Bigg)$ on the right hand side of (6), we see that, given (8), (6) is satisfied if

$$0 \geq C\Bigg(x\left(\hat{\theta},\hat{\eta}\right), \hat{\eta}, \hat{\theta}\Bigg) - C\Bigg(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right), \hat{\theta}\Bigg) - \Bigg(C\Bigg(x\left(\hat{\theta},\hat{\eta}\right), \hat{\eta}, \theta\Bigg) - C\Bigg(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right), \theta\Bigg)\Bigg).$$

Using $C(x,q,\theta) = c(x,\theta) + k(x,q)$, we can write this as

$$0 \geq k\left(x\left(\hat{\theta},\hat{\eta}\right), \hat{\eta}\right) - k\left(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right) - \left(k\left(x\left(\hat{\theta},\hat{\eta}\right), \hat{\eta}\right) - k\left(x\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right)\right) = 0$$

so we have established incentive compatibility of $\tilde{x}\left(\hat{\theta},\hat{\eta}\right)$, $\tilde{q}\left(\hat{\theta},\hat{\eta}\right)$, and $\tilde{t}\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$.

Consider now the surplus. The agent's equilibrium payoffs are unchanged, as

$$\begin{aligned} & t(\theta,\eta) - C(x(\theta,\eta), q(\theta,\eta), \theta) \\ = \quad & t(\theta,\eta) + C(x(\theta,\eta), \eta, \theta) - C(x(\theta,\eta), q(\theta,\eta), \theta) - C(x(\theta,\eta), \eta, \theta). \end{aligned}$$

In other words, the agent is just compensated for the increase in his cost of production, and the entire additional surplus goes to the principal. But in this case Proposition 1 shows that higher quality is desirable. ■

Additive separability of the cost function is crucial for this result. If the cost function is not additively separable in θ and q, then changing the allocation will change the agent's incentive to mimic other types. In this case a proof as simple as ours would not establish the desirability of higher quality in the second best; alternative assumptions and techniques of proof would be required. Without mentioning them further, the original regularity conditions plus additive separability of the cost function in θ and q are maintained throughout the paper.

Consider next the incentive constraint (2). The number of potential deviations to consider is very large—a crucial difficulty that problems of multidimensional screening face. However, this problem has properties that permit a substantial reduction in dimensionality.

Proposition 3 *The incentive constraints* (2)*are satisfied if and only if the constraints*

$$t(\theta,\eta) - C(x(\theta,\eta),\eta,\theta) \geq t\left(\hat{\theta},\eta\right) - C\left(x\left(\hat{\theta},\eta\right),\eta,\theta\right) \text{ for all } \hat{\theta} \tag{9}$$

and

$$t(\theta,\eta) - C(x(\theta,\eta),\eta,\theta) \geq t(\theta,\hat{\eta}) - C(x(\theta,\hat{\eta}),\hat{\eta},\theta) \text{ for all } \hat{\eta} \leq \eta \tag{10}$$

are satisfied.

Proof. It is easy to see that the two one dimensional constraints are necessary for incentive compatibility. We now show they are also sufficient for incentive compatibility. If a type (θ,η) mimics type $\left(\hat{\theta},\hat{\eta}\right)$, he obtains utility $t\left(\hat{\theta},\hat{\eta}\right) - C\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\eta},\theta\right)$. But by incentive compatibility of type $(\theta,\hat{\eta})$ in the θ dimension, type (θ,η) could obtain more by mimicking type $(\theta,\hat{\eta})$, since

$$t(\theta,\hat{\eta}) - C(x(\theta,\hat{\eta}),\hat{\eta},\theta) \geq t\left(\hat{\theta},\hat{\eta}\right) - C\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\eta},\theta\right)$$

But then, by incentive compatibility of type (θ,η) in the η dimension, we have

$$t(\theta,\eta) - C(x(\theta,\eta),\eta,\theta) \geq t(\theta,\hat{\eta}) - C(x(\theta,\hat{\eta}),\hat{\eta},\theta) \text{ for all } \hat{\eta} \leq \eta$$

showing that there cannot be a profitable deviation. ■

The essential feature that drives this result is that the agent's payoff only depends on his announced ability to produce quality, $\hat{\eta}$, but not on η itself. Therefore, if a type (θ,η) mimics a type $(\theta,\hat{\eta})$ with $\hat{\eta} \leq \eta$, he obtains exactly the same utility that type $(\theta,\hat{\eta})$ would obtain. This is a crucial difference to the general case of multidimensional screening and greatly simplifies the analysis. Instead of solving the problem, (1) subject to (2) and (3), one can solve the problem (1) subject to (3), (9), and (10) in which only the one dimensional incentive constraints are relevant. However, this problem is still not easy to solve, because there are still substantially more constraints than in the usual one-dimensional screening problem. Our strategy to solve this problem is to build further on the result that only one-dimensional deviations have to be ruled out. In particular, we show that the principal's problem can be solved by screening the θ types conditional on η. The reason is, that under reasonable conditions on the joint distribution of types, the constraint (10) is not binding at the optimum. For this purpose, it is useful to first find a solution to the problem when only θ is unobservable and η is common knowledge.

2.3 The Case of Observable Quality Bounds

In this case, the buyer can condition on η. We let $x(\theta;\eta)$ and $t(\theta;\eta)$ for all θ denote the quantity and payment schedule conditional on η. The buyer solves, for each given η, the following problem

$$\max_{x(\cdot;\eta),t(\cdot;\eta)} \int_{\underline{\theta}}^{\overline{\theta}} \left(V\left(x\left(\theta;\eta\right),\eta\right) - t\left(\theta;\eta\right)\right) f\left(\theta|\,\eta\right) d\theta \tag{11}$$

subject to, for all θ, η :

$$t\left(\theta;\eta\right) - C\left(x\left(\theta;\eta\right),\eta,\theta\right) \geq t\left(\hat{\theta};\eta\right) - C\left(x\left(\hat{\theta};\eta\right),\eta,\theta\right) \text{ for all } \hat{\theta} \quad \text{and} \tag{12}$$

$$t\left(\theta;\eta\right) - C\left(x\left(\theta;\eta\right),\eta,\theta\right) \geq 0. \tag{13}$$

This is a standard problem and is normally solved by reformulating the incentive and participation constraints. We state a more tractable version of these constraints in the following lemma. We call a pair of quantity schedule $x(\theta;\eta)$ and payment schedule $t(\theta;\eta)$ implementable if they satisfy constraints (12) and (13).

Lemma 1 *The pair of quantity schedule $x(\theta;\eta)$ and payment schedule $t(\theta;\eta)$ is implementable if and only if*

$$t\left(\theta;\eta\right) = C\left(x\left(\theta;\eta\right),\eta,\theta\right) + \int_{\theta}^{\overline{\theta}} C_y\left(x\left(y;\eta\right),\eta,y\right) dy \tag{14}$$

and $x(\theta;\eta)$ is non-increasing in θ.

Proof. Let $u(\theta;\eta) = \max_{\hat{\theta}} t\left(\hat{\theta};\eta\right) - C\left(x\left(\hat{\theta};\eta\right),\eta,\theta\right)$. Then, by the envelope theorem,

$$u_\theta\left(\theta;\eta\right) = -C_\theta\left(x\left(\theta;\eta\right),\eta,\theta\right)$$

Since $u_\theta(\theta;\eta) < 0$, the participation constraint must be binding for type $\overline{\theta}$, so $u\left(\overline{\theta};\eta\right) = 0$ Thus, we can write

$$u\left(\theta;\eta\right) = \int_{\theta}^{\overline{\theta}} C_y\left(x\left(y;\eta\right),\eta,y\right) dy$$

Since $u(\theta;\eta) = t(\theta;\eta) - C(x(\theta;\eta),\eta,\theta)$, we can write

$$t(\theta;\eta) = C(x(\theta;\eta),\eta,\theta) + \int_{\theta}^{\bar{\theta}} C_y(x(y;\eta),\eta,y)\,dy.$$

It is then standard to show that this pair of quantity and payment schedules satisfies global incentive compatibility if $x(\theta;\eta)$ is non-increasing in θ. This is omitted. ■

Substituting the transfers into the principal's objective function and integrating by parts, we can write the principal's problem as

$$\max_{x(\cdot;\eta)} \int_{\underline{\theta}}^{\bar{\theta}} \left(V(x(\theta;\eta),\eta) - C(x(\theta;\eta),\eta,\theta) - C_\theta(x(\theta;\eta),\eta,\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)} \right) f(\theta|\eta)\,d\theta$$

subject to $x(\theta;\eta)$ being non-increasing in θ.

It is customary to solve this problem imposing a regularity condition on the distribution of types that ensures that the constraint is never binding at the solution to this problem as well as additional assumptions on the cost function.

Proposition 4 *Assume that $C_{xx\theta}(x,\eta,\theta) \geq 0$, $C_{x\theta\theta}(x,\eta,\theta) \geq 0$, and $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is non-decreasing in θ. Then, the optimal quantity schedule satisfies the first-order condition*

$$V_x(x(\theta;\eta),\eta) = C_x(x(\theta;\eta),\eta,\theta) + C_{x\theta}(x(\theta;\eta),\eta,\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)}. \tag{15}$$

Proof. Differentiating totally, we have

$$\frac{dx}{d\theta} = \frac{\left(C_{x\theta}(x(\theta;\eta),\eta,\theta) + C_{x\theta\theta}(x(\theta;\eta),\eta,\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)} + C_{x\theta}(x(\theta;\eta),\eta,\theta)\frac{\partial}{\partial\theta}\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}{\left(V_{xx}(x(\theta;\eta),\eta) - C_{xx}(x(\theta;\eta),\eta,\theta) - C_{xx\theta}(x(\theta;\eta),\eta,\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}$$

The denominator is a second-order condition to the principal's problem, and the assumption $C_{xx\theta}(x,\eta,\theta) \geq 0$ ensures that it is satisfied. The numerator is non-negative by our assumptions on the cost function and the monotonicity of the inverse hazard function, thus guaranteeing that $x(\theta;\eta)$ is non decreasing in θ. ■

Since both the procedure to solve this one-dimensional problem and the solution that emerges from it are well known (see, e.g., Fudenberg and Tirole (1991)) we do not dwell on the details. However, for our further results it is useful to review the economics behind the distortions inherent in (15). We can write the first-order condition as

$$\Big(V_x\left(x\left(\theta;\eta\right),\eta\right)-C_x\left(x\left(\theta;\eta\right),\eta,\theta\right)\Big)f\left(\theta|\,\eta\right)=C_{x\theta}\left(x\left(\theta;\eta\right),\eta,\theta\right)F\left(\theta|\,\eta\right)$$

On the left side we have the principal's desire to implement an efficient solution, which would require that the marginal benefit of consumption to the principal is equal to the marginal cost of production to the agent. The weight given to this motive is $f\left(\theta|\,\eta\right)$, the likelihood of type $\theta|\,\eta$. On the right side appears the principal's desire to limit the agent's rents. An increase in $x\left(\theta;\eta\right)$ increases the rents that have to be given to all types that are more efficient at producing than type $\theta|\,\eta$. Since there is a mass $F\left(\theta|\,\eta\right)$ of these people, the weight attached to this motive is $F\left(\theta|\,\eta\right)$.

Next, we show that, remarkably, the solution to this problem is also a solution to the overall problem, provided that the trade-off between efficiency and rent extraction is affected in the "right way" by changes in η. We address this problem in the next subsection.

2.4 The Case of Unknown abilities and costs

We now show that, given certain restrictions on the joint distribution of characteristics, the agent has no incentive to mimic another type who produces a lower quality level. Formally, we have the following result:

Proposition 5 *If, in addition to the assumptions of Proposition 4, η and θ are affiliated, then unobservability of η does not affect the solution to the principal's problem. Formally, the solution is still given by the quantity schedule* (15) *and the payment schedule* (14).

Proof. Suppose the principal offers quantity schedule (15) and payment schedule (14). Thus, we identify the schedules $x\left(\theta,\eta\right)\equiv x\left(\theta;\eta\right)$ and $t\left(\theta,\eta\right)\equiv t\left(\theta;\eta\right)$ for all θ. If the agent announces his true quality η, then his indirect utility is

$$u\left(\theta,\eta\right)=\int_{\theta}^{\overline{\theta}}C_y\left(x\left(y,\eta\right),\eta,y\right)dy$$

If he under reports $\hat{\eta}$, then he obtains indirect utility

$$u\left(\theta,\hat{\eta}\right)=\int_{\theta}^{\overline{\theta}}C_y\left(x\left(y,\hat{\eta}\right),\hat{\eta},y\right)dy$$

So, we need to show that $u(\theta,\eta)$ is non-decreasing in η. This will be the case when the integrand is non-decreasing in η for each y and θ. Differentiating under the integral, we obtain

$$C_{\theta x}\left(x\left(\theta,\eta\right),\eta,\theta\right)\frac{dx}{d\eta}+C_{\theta\eta}\left(x\left(\theta,\eta\right),\eta,\theta\right)$$

From our specification of additive separability, we have $C_{\theta\eta}\left(x\left(\theta,\eta\right),\eta,\theta\right)=0$. So, we only need to show that $\frac{dx}{d\eta}\geq 0$. Differentiating (15) again totally around the stationary point, (and using again that $C_{x\theta\eta}\left(x\left(\theta,\eta\right),\eta,\theta\right)=0$) we have

$$\frac{dx}{d\eta}=\frac{\left(-V_{x\eta}\left(x\left(\theta,\eta\right),\eta\right)+C_{x\eta}\left(x\left(\theta,\eta\right),\eta,\theta\right)+C_{x\theta}\left(x\left(\theta,\eta\right),\eta,\theta\right)\frac{\partial}{\partial\eta}\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}{\left(V_{xx}\left(x\left(\theta,\eta\right),\eta\right)-C_{xx}\left(x\left(\theta,\eta\right),\eta,\theta\right)-C_{xx\theta}\left(x\left(\theta,\eta\right),\eta,\theta\right)\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}.$$

By assumption we have $V_{x\eta}\left(x\left(\theta,\eta\right),\eta\right)-C_{x\eta}\left(x\left(\theta,\eta\right),\eta,\theta\right)\geq 0$. Thus, to complete the proof, we show that affiliation implies that $\frac{\partial}{\partial\eta}\frac{F(\theta|\eta)}{f(\theta|\eta)}\leq 0$.

Recall that two random variables are affiliated if for $\theta\geq\theta'$ and $\eta\geq\eta'$

$$f\left(\theta,\eta\right)f\left(\theta',\eta'\right)\geq f\left(\theta',\eta\right)f\left(\theta,\eta'\right).$$

Dividing on both sides by $g\left(\eta\right)g\left(\eta'\right)$, where $g\left(\eta\right)=\int_{\underline{\theta}}^{\overline{\theta}}f\left(\theta,\eta\right)d\theta$ and $g\left(\eta'\right)=\int_{\underline{\theta}}^{\overline{\theta}}f\left(\theta,\eta'\right)d\theta$ are the marginal densities, we can write

$$f\left(\theta\left|\eta\right.\right)f\left(\theta'\left|\eta'\right.\right)\geq f\left(\theta'\left|\eta\right.\right)f\left(\theta\left|\eta'\right.\right).$$

Integrating over θ' between $\underline{\theta}$ and θ we find

$$f\left(\theta\left|\eta\right.\right)F\left(\theta\left|\eta'\right.\right)\geq F\left(\theta\left|\eta\right.\right)f\left(\theta\left|\eta'\right.\right).$$

Rearranging, we have

$$\frac{F\left(\theta\left|\eta'\right.\right)}{f\left(\theta\left|\eta'\right.\right)}\geq\frac{F\left(\theta\left|\eta\right.\right)}{f\left(\theta\left|\eta\right.\right)}$$

which is, if $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is differentiable in η, equivalent to $\frac{\partial}{\partial\eta}\left(\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)\leq 0$. ■

The intuition behind this result is straightforward. Recall that the indirect utility of a type $\theta\left|\eta\right.$ is given by $u\left(\theta;\eta\right)=\int_{\theta}^{\overline{\theta}}C_y\left(x\left(y;\eta\right),\eta,y\right)dy$. The rent of this type is determined by the production schedule offered to all types who are less efficient than this type. So, type $\theta\left|\eta\right.$ has no incentive to mimic a type of a lower quality $\hat{\eta}<\eta$ if less able types produce smaller quantities. We can now understand why this will indeed be the case provided that types are affiliated.

The higher is η the higher is $\frac{f(\theta|\eta)}{F(\theta|\eta)}$, so the greater is the weight given to the principal's efficiency motive as opposed to the motive to limit the agent's rents. Put another way, it is relatively less likely that the agent is a low cost producer when he produces a high level of quality. Therefore, the rent given to any given type $\theta\,|\eta$ is higher the higher is η, and this type has no incentive to report a lower value of his quality parameter η.

3 Regulation

The regulation problem was first studied by Baron and Myerson [1982] when the firm's costs are unknown to the regulator. Lewis and Sappington [1988] have extended their analysis when in addition to cost, demand conditions - more precisely the intercept of a linear demand function - are unobservable to the regulator.[5] We provide a variant on this two-dimensional screening problem where the level of demand is affected by a quality choice made by the firm and the regulator can observe the quality choice but not the upper bound on the quality the producer is able to provide. We show that this two-dimensional screening problem can be solved using the same techniques as in the procurement problem in Section 2.

Consumers' valuations for a quantity x of a good whose quality is q are described by the downward sloping inverse demand function $P(x,q)$. Define the gross consumer surplus of a consumer who buys x units of a good of quality q at a constant marginal price as

$$V(x,q) \equiv \int_0^x P(z,q)\,dz.$$

As in Section 2, the good is produced by a firm with a cost of production of $C(x,q,\theta)$ for $q \leq \eta$ and infinity otherwise.

The regulator maximizes a weighted sum of net consumer surplus and producer surplus under incentive constraints. The regulator's instruments are a constant payment t, a marginal price p, and a choice of quality q. Under a

[5] See Armstrong [1999] for an analysis of some technical problems in Lewis and Sappington.

truthful mechanism, the joint surplus for a given tuple (θ, η) is equal to

$$V\Big(X\big(p(\theta,\eta), q(\theta,\eta)\big)\Big) - p(\theta,\eta) X\big(p(\theta,\eta), q(\theta,\eta)\big) - t(\theta,\eta)$$
$$+\alpha\left(t(\theta,\eta) + p(\theta,\eta) X\big(p(\theta,\eta), q(\theta,\eta)\big) - C\Big(X\big(p(\theta,\eta), q(\theta,\eta)\big), q(\theta,\eta), \theta\Big)\right)$$

where $X(p, q)$ is the direct demand function for the good which, is assumed to be downward sloping, $X_p(p, q) < 0$.

It is easy to see that Propositions 1, 2 and 3 apply to this model. The idea is to hold the profit of the firm constant by compensating the firm exactly for increases in costs due to an increase in the quality the firm produces. That means that the change in $t(\theta, \eta)$ due to a slight increase in q is just equal to $C_q(x(\theta, \eta), q(\theta, \eta), \theta)$; the increase in gross consumer surplus is $V_q(x(\theta, \eta), q(\theta, \eta))$. Hence, under our assumptions it is optimal to produce the highest quality, so $q(\theta, \eta) = \eta$ for all θ, η.

Proceeding the same way as in the procurement model, we solve the screening problems conditional on η and prove the incentive compatibility of these problems' solutions using the solutions themselves. Let

$$\pi(\theta; \eta) = \max_{\hat{\theta}} \left\{ t\left(\hat{\theta}; \eta\right) + p\left(\hat{\theta}; \eta\right) X\left(p\left(\hat{\theta}; \eta\right), \eta\right) - C\left(X\left(p\left(\hat{\theta}, \eta\right), \eta\right), \eta, \theta\right) \right\}.$$

where we use again the notation $(\theta; \eta)$ to indicate that we condition on η.

Lemma 2 *The price and payment schedules, $p(\theta; \eta)$ and $t(\theta; \eta)$, are implementable if and only if*

$$\pi(\theta; \eta) = \int_{\theta}^{\overline{\theta}} C_z(X(p(z; \eta), \eta), \eta, z)\, dz \qquad (16)$$

and $p(\theta; \eta)$ is non-decreasing in θ.

Proof. The proof of (16) proceeds just as in the proof of Lemma 1. We now show that the price schedule must be non-decreasing.

Incentive compatibility requires that type $(\theta; \eta)$ has no incentive to mimic type $\left(\hat{\theta}; \eta\right)$

$$t(\theta; \eta) + p(\theta; \eta) X(p(\theta; \eta), \eta) - C(X(p(\theta; \eta), \eta), \eta, \theta)$$
$$\geq t\left(\hat{\theta}; \eta\right) + p\left(\hat{\theta}; \eta\right) X\left(p\left(\hat{\theta}; \eta\right), \eta\right) - C\left(X\left(p\left(\hat{\theta}; \eta\right), \eta\right), \eta, \theta\right)$$

and that type $\left(\hat{\theta};\eta\right)$ has no incentive to mimic type $(\theta;\eta)$

$$t\left(\hat{\theta};\eta\right)+p\left(\hat{\theta};\eta\right)X\left(p\left(\hat{\theta};\eta\right),\eta\right)-C\left(X\left(p\left(\hat{\theta};\eta\right),\eta\right),\eta,\hat{\theta}\right)$$
$$\geq t\left(\theta;\eta\right)+p\left(\theta;\eta\right)X\left(p\left(\theta;\eta\right),\eta\right)-C\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\hat{\theta}\right).$$

Summing these inequalities gives

$$C\left(X\left(p\left(\hat{\theta};\eta\right),\eta\right),\eta,\theta\right)-C\left(X\left(p\left(\hat{\theta};\eta\right),\eta\right),\eta,\hat{\theta}\right)$$
$$\geq C\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\theta\right)-C\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\hat{\theta}\right).$$

Writing as integrals, we have

$$\int_{\hat{\theta}}^{\theta}C_z\left(X\left(p\left(\hat{\theta};\eta\right),\eta\right),\eta,z\right)dz\geq\int_{\hat{\theta}}^{\theta}C_z\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,z\right)dz.$$

Given that $C_{x\theta}\geq 0$ this requires for $\theta>\hat{\theta}$ that $X\left(p\left(\hat{\theta};\eta\right),\eta\right)\geq X\left(p\left(\theta;\eta\right),\eta\right)$ or $p\left(\hat{\theta};\eta\right)\leq p\left(\theta;\eta\right)$. Hence, prices must be non-decreasing in θ. ■

Substituting $\pi\left(\theta;\eta\right)$ into the objective function, and integrating by parts, we can write the regulator's problem as

$$\int_{\underline{\theta}}^{\overline{\theta}}\left[V\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta\right)-C\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\theta\right)\right]f\left(\theta\left|\eta\right.\right)d\theta$$
$$-\int_{\underline{\theta}}^{\overline{\theta}}\left[\left(1-\alpha\right)C_\theta\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\theta\right)\frac{F\left(\theta\left|\eta\right.\right)}{f\left(\theta\left|\eta\right.\right)}\right]f\left(\theta\left|\eta\right.\right)d\theta$$

subject to $p\left(\theta;\eta\right)$ being non-decreasing in θ.

Proposition 6 *The optimum satisfies the first-order condition*

$$\left(p\left(\theta;\eta\right)-C_x\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\theta\right)-\left(1-\alpha\right)C_{x\theta}\left(X\left(p\left(\theta;\eta\right),\eta\right),\eta,\theta\right)\frac{F\left(\theta\left|\eta\right.\right)}{f\left(\theta\left|\eta\right.\right)}\right)X_p\left(p\left(\theta;\eta\right)\right)=0.$$

Proof. Recalling that the cost function can be written as

$$C(c,q,\theta)=c(x,\theta)+k(x,q),$$

the first-order condition simplifies to

$$p(\theta;\eta) - c_x(X(p(\theta;\eta)),\theta) - (1-\alpha)\, c_{x\theta}(X(p(\theta;\eta)),\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)} = 0.$$

Differentiating totally with respect to θ and p we have

$$\frac{dp}{d\theta} = \frac{\left(-c_{x\theta}(X(p(\theta;\eta)),\theta) - (1-\alpha)\, c_{x\theta\theta}(X(p(\theta;\eta)),\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)} - (1-\alpha)\, c_{x\theta}(X(p(\theta;\eta)),\theta)\frac{\partial}{\partial\theta}\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}{-\left(1-\left(c_{xx}(X(p(\theta;\eta)),\theta) + (1-\alpha)\, c_{xx\theta}(X(p(\theta;\eta)),\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)}\right) X_p(p(\theta;\eta))\right)}.$$

The denominator of this expression is negative by the second-order condition of the regulator's maximization problem. Using our assumption that $c_{x\theta} \geq 0$, $c_{x\theta\theta} \geq 0$, and $\frac{\partial}{\partial\theta}\frac{F(\theta|\eta)}{f(\theta|\eta)} \geq 0$, we observe that $\frac{dp}{d\theta} \geq 0$ as required for incentive compatibility. ■

Next, we show that the solution remains incentive compatible when η is no longer observable.

Proposition 7 *Given the regularity conditions of Propositions 4 and 5, the unobservability of η does not affect the solution to the principal's problem.*

Proof. Again, we identify the schedules $p(\theta;\eta) \equiv p(\theta,\eta)$ and $\pi(\theta;\eta) \equiv \pi(\theta,\eta)$ for all θ. Using the additive separability of the cost function and differentiating the profit of the firm with respect to η we have

$$\pi_\eta(\theta,\eta) = \int_\theta^{\overline{\theta}} \left[c_{zx}(X(p(z,\eta)),z)\, X_p(p(z,\eta))\frac{dp(z,\eta)}{d\eta}\right] dz.$$

A sufficient condition for incentive compatibility is $\frac{dp(z,\eta)}{d\eta} \leq 0$. Differentiating the first-order condition totally with respect to η and p we have

$$\frac{dp}{d\eta} = \frac{\left(-(1-\alpha)\, c_{x\theta}(X(p(\theta,\eta)),\theta)\frac{\partial}{\partial\eta}\frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}{-\left(1-\left(c_{xx}(X(p(\theta,\eta)),\theta) + (1-\alpha)\, c_{xx\theta}(X(p(\theta,\eta)),\theta)\frac{F(\theta|\eta)}{f(\theta|\eta)}\right) X_p(p(\theta,\eta))\right)}.$$

Since the denominator is negative, $\frac{dp(z,\eta)}{d\eta} \leq 0$ if $\frac{\partial}{\partial\eta}\frac{F(\theta|\eta)}{f(\theta|\eta)} \leq 0$. This is exactly what affiliation implies. ■

Thus, we have shown that all the results of Section 2 apply to the regulation application as well as to the procurement problem discussed there.

4 Selling to a budget constrained buyer

Che and Gale (2000) investigate the problem of selling to a buyer who has private information about his valuation of a good and about how much he can spend on the good. In other words, the buyer faces a budget constraint and the seller's choice of selling procedure may be constrained by the fact that the buyer cannot spend more than he has. Che and Gale distinguish between the case where the seller is able to demand that the buyer place a bond and the case where the seller is not able to do so. If the seller can demand the placement of such a bond, then, the buyer is effectively only able to mimic types with a lower budget, but not types with a higher budget, because in that case the buyer would be unable to pay. We treat the case where the seller can require the buyer to place a bond because it is readily amenable to the approach proposed in Section 2.

A buyer's type is a tuple θ, η (v, w in Che and Gale's notation) where θ is the valuation of a buyer for a good offered by the seller and η is the buyer's spending limit. The seller has one unit of a good on sale. We let $x(\theta, \eta)$ denote the probability of delivery of the good to the buyer of type (θ, η). Equivalently, we may think of $x(\theta, \eta)$ as the fraction of the good delivered to the buyer, in case the good is actually divisible. We let $t(\theta, \eta)$ denote the payment made by the buyer with type (θ, η). The budget constraint of the buyer has the following implications. First, any equilibrium payment must satisfy the feasibility constraint

$$t(\theta, \eta) \leq \eta.$$

Second, the budget constraint limits the buyer's ability to mimic other types. Given the feasibility constraint, and the placement of the bond that rules out exaggerating the budget η, the incentive constraint of a type (θ, η) takes the form that for all (θ, η)

$$\theta x(\theta, \eta) - t(\theta, \eta) \geq \theta x\left(\hat{\theta}, \hat{\eta}\right) - t\left(\hat{\theta}, \hat{\eta}\right) \text{ for all } \hat{\theta}, \hat{\eta} \leq \eta. \tag{17}$$

Finally, all types must be willing to particpate, that is

$$\theta x(\theta, \eta) - t(\theta, \eta) \geq 0.^{6} \tag{18}$$

[6] The actual utility of the agent is $\eta + \theta x(\theta, \eta) - t(\theta, \eta)$ so that the participation constraint is written as $\eta + \theta x(\theta, \eta) - t(\theta, \eta) \geq \eta$ and similarly for the incentive constraints (17). Because of the additivity in η, solving the problem in the net utility from participation, $\theta x(\theta, \eta) - t(\theta, \eta)$, is sufficient.

The seller maximizes

$$\int_{\underline{\eta}}^{\overline{\eta}} \int_{\underline{\theta}}^{\overline{\theta}} t(\theta, \eta) f(\theta, \eta) \, d\theta d\eta$$

subject to constraints (17) and (18).

We solve this problem as before. We begin with the problem where the seller knows η and conditions the contracts on η. Then, we show that no buyer has an incentive to mimic any type with a lower budget.

If the seller knows that the buyer has a budget of η, then she solves the following problem:

$$\max_{x(\cdot;\eta), t(\cdot;\eta)} \int_{\underline{\theta}}^{\overline{\theta}} (t(\theta; \eta)) f(\theta | \eta) \, d\theta$$

such that for all θ

$$\begin{aligned} \theta x(\theta; \eta) - t(\theta; \eta) &\geq \theta x\left(\hat{\theta}; \eta\right) - t\left(\hat{\theta}; \eta\right) \text{ for all } \hat{\theta}, \\ \theta x(\theta; \eta) - t(\theta; \eta) &\geq 0, \quad \text{and} \\ t(\theta; \eta) &\leq \eta. \end{aligned}$$

Letting the net indirect utility gain of type (θ, η) be

$$u(\theta; \eta) = \max_{\hat{\theta}} \left(\hat{\theta}; \eta\right) - t\left(\hat{\theta}; \eta\right)$$

we know from the argument in Section 2 that it can be written as

$$u(\theta; \eta) = u(\underline{\theta}; \eta) + \int_{\underline{\theta}}^{\theta} x(z; \eta) \, dz.$$

Clearly it is optimal to extract all the rents from the type with the lowest valuation, so $u(\underline{\theta}; \eta) = 0$. Using the fact that

$$u(\theta; \eta) = \theta x(\theta; \eta) - t(\theta; \eta)$$

we can substitute

$$t(\theta; \eta) = \theta x(\theta; \eta) - \int_{\underline{\theta}}^{\theta} x(z; \eta) \, dz \tag{19}$$

into the seller's objective function. After an integration by parts, it can be written

$$\max_{x(\cdot,\eta)} \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta|\,\eta)}{f(\theta|\,\eta)}\right) x(\theta;\eta)\, f(\theta|\,\eta)\, d\theta$$

subject to

$$x(\theta;\eta) \text{ non-decreasing in } \theta, \text{ and}$$

$$\theta x(\theta;\eta) - \int_{\underline{\theta}}^{\theta} x(z;\eta)\, dz \leq \eta.$$

Given the following assumptions on the joint distribution of characteristics, our pointwise procedure applies: assume that for each η the distribution of θ conditional on η satisfies

$$\frac{\partial}{\partial\theta}\frac{1-F(\theta|\,\eta)}{f(\theta|\,\eta)} \leq 0 \tag{20}$$

and that the random variables $(\theta, -\eta)$ are affiliated, implying that

$$\frac{\partial}{\partial\eta}\frac{1-F(\theta|\,\eta)}{f(\theta|\,\eta)} \leq 0. \tag{21}$$

Let $\theta^*(\eta)$ solve the equation

$$\theta^*(\eta) = \frac{1-F(\theta^*(\eta)|\,\eta)}{f(\theta^*(\eta)|\,\eta)}. \tag{22}$$

By (20), there is at most one such value. Assuming, in addition, that $\underline{\theta} \geq \frac{1}{f(\underline{\theta}|\eta)}$, implies there is exactly one value $\theta^*(\eta)$. The objective is increasing in $x(\theta;\eta)$ for $\theta < \theta^*(\eta)$, and decreasing in $x(\theta;\eta)$ for $\theta > \theta^*(\eta)$. The solution depends on whether $\theta^*(\eta) \leq \eta$ or $\theta^*(\eta) > \eta$.

In the former case, when $\theta^*(\eta) \leq \eta$, the budget constraint of the buyer is slack when the seller implements the optimal selling mechanism. To see this, suppose that the the budget constraint is indeed slack. Then, the solution involves $x^*(\theta;\eta) = 0$ for $\theta < \theta^*(\eta)$ and $x^*(\theta;\eta) = 1$ for $\theta \geq \theta^*(\eta)$. The payment then takes the form of

$$t(\theta;\eta) = \begin{cases} 0 & \text{for } \theta < \theta^*(\eta) \\ \theta^*(\eta) & \text{for } \theta \geq \theta^*(\eta). \end{cases}$$

Clearly, this payment schedule satisfies the budget constraint for all buyers only if $\theta^*(\eta) \leq \eta$.

Suppose, on the other hand, that $\theta^*(\eta) > \eta$. Then, the budget constraint must be binding for some types and the presence of the term $\int_{\underline{\theta}}^{\theta} x(z;\eta)\,dz$ in the budget constraint forces us to use control techniques. Let $y(\theta;\eta) \equiv \int_{\underline{\theta}}^{\theta} x(z;\eta)\,dz$ and consider the optimal control problem with

$$H(\theta;\eta) = \left(\theta - \frac{1-F(\theta|\,\eta)}{f(\theta|\,\eta)}\right) f(\theta|\,\eta)\, x(\theta;\eta)$$

subject to

$$y_\theta(\theta;\eta) = x(\theta;\eta)$$

and

$$\theta x(\theta;\eta) - y(\theta;\eta) \leq \eta.$$

The Lagrangian for this problem is

$$L = \left(\theta - \frac{1-F(\theta|\,\eta)}{f(\theta|\,\eta)}\right) x(\theta;\eta)\, f(\theta|\,\eta) + \lambda(\theta)\, x(\theta;\eta) + \mu(\theta)\,(\eta - \theta x(\theta;\eta) + y(\theta;\eta)).$$

From the maximum principle, the optimal value of the control variable, $x^*(\theta;\eta)$, satisfies

$$x^*(\theta;\eta) = \arg\max_{x(\theta;\eta)} \left(\left(\theta - \frac{1-F(\theta|\,\eta)}{f(\theta|\,\eta)}\right) f(\theta|\,\eta) + \lambda(\theta) - \mu(\theta)\,\theta\right) x(\theta;\eta),$$

$\mu(\theta) \geq 0$, $\eta - \theta x(\theta;\eta) + y(\theta;\eta) \geq 0$, and

$$\mu(\theta)\,(\eta - \theta x(\theta;\eta) + y(\theta;\eta)) = 0.$$

Moreover, the optimal path of the costate variable, $\lambda(\theta)$, satisfies the conditions

$$\lambda_\theta(\theta) = -\frac{\partial L}{\partial y(\theta,\eta)} = -\mu(\theta),$$

and

$$\lambda(\overline{\theta}) = 0$$

where the latter condition is a transversality condition we impose on our problem with given initial value $y(\underline{\theta};\eta) = 0$ and free endpoint $y(\overline{\theta};\eta)$.

The solution to this problem is of the form that $x^*(\theta;\eta) = 0$ for $\theta < \theta^*(\eta)$ and $x^*(\theta;\eta) = x(\eta)$ for $\theta \geq \theta^*(\eta)$ where $x(\eta)$ solves

$$\theta^*(\eta)\, x(\eta) = \eta.$$

The associated payment schedule is $t(\theta;\eta) = 0$ for $\theta < \theta^*(\eta)$ and $t(\theta;\eta) = \eta$ for $\theta \geq \theta^*(\eta)$.

To see this, note first that for $\theta < \theta^*(\eta)$, L is decreasing in $x(\theta;\eta)$ because the term $\left(\theta - \frac{1-F(\theta|\eta)}{f(\theta|\eta)}\right) x(\theta;\eta) f(\theta|\eta)$ is decreasing in $x(\theta;\eta)$. Hence the seller does not want to sell to these types, which implies that $\mu(\theta) = 0$ for $\theta < \theta^*(\eta)$. Since the state variable $y(\theta;\eta)$ enters the problem exclusively through the binding budget constraint, the sellers objective function for $\theta < \theta^*(\eta)$ is independent of the value of $y(\theta;\eta)$ so that $\lambda(\theta) = 0$ for $\theta < \theta^*(\eta)$.

For $\theta \geq \theta^*(\eta)$, the term $\left(\theta - \frac{1-F(\theta|\eta)}{f(\theta|\eta)}\right) x(\theta;\eta) f(\theta|\eta)$ is increasing in $x(\theta;\eta)$. Hence, if the budget constraint of the buyer is non-binding, then the seller would like to set $x(\theta;\eta) = 1$ for $\theta \geq \theta^*(\eta)$. For the sake of the argument, suppose the seller already sets $x(\theta;\eta) = 1$ for $\theta = \theta^*(\eta)$. But then the payment made by type $\theta^*(\eta)$ would have to satisfy $\theta^*(\eta) - y(\theta^*(\eta);\eta) \leq \eta$. But since $y(\theta^*(\eta);\eta) = 0$ and $\theta^*(\eta) > \eta$, this condition is violated.

Finally, it remains to be shown that the budget constraint must be binding for all types larger than $\theta^*(\eta)$. The reason is that incentive compatibility implies that the payment is non-decreasing in θ. Differentiating (19) with respect to θ we obtain

$$t_\theta(\theta;\eta) = \theta x_\theta(\theta;\eta) + x(\theta;\eta) - x(\theta;\eta) = \theta x_\theta(\theta;\eta) \geq 0$$

In fact the binding budget constraint directly implies that $x_\theta(\theta;\eta) = 0$; if the budget constraint is binding over an interval then the change of expenditure, $t_\theta(\theta;\eta)$, must equal the change of the budget, 0, over that interval. Hence, $x_\theta(\theta;\eta) = 0$ over the entire interval $\left[\theta^*(\eta), \overline{\theta}\right]$. Hence $x(\theta;\eta)$ is equal to a constant, $x(\eta)$ for all $\theta \in \left[\theta^*(\eta), \overline{\theta}\right]$.

Next we show that the solution satisfies incentive compatibility in the η dimension. To accomplish this, we have to show that $x(\theta;\eta)$ is non-decreasing in η.

First, we show that the value of $\theta^*(\eta)$ is non-increasing in η. Totally differentiating (22) yields

$$\left(1 - \frac{\partial}{\partial\theta^*}\frac{1-F(\theta^*(\eta)|\eta)}{f(\theta^*(\eta)|\eta)}\right) d\theta^* = \frac{\partial}{\partial\eta}\frac{1-F(\theta^*(\eta)|\eta)}{f(\theta^*(\eta)|\eta)} d\eta$$

so that

$$\frac{d\theta^*}{d\eta} = \frac{\frac{\partial}{\partial\eta}\frac{1-F(\theta^*(\eta)|\eta)}{f(\theta^*(\eta)|\eta)}}{\left(1 - \frac{\partial}{\partial\theta^*}\frac{1-F(\theta^*(\eta)|\eta)}{f(\theta^*(\eta)|\eta)}\right)}.$$

Note that (20) and (21) imply that $\frac{d\theta^*}{d\eta} \leq 0$.

Second, we show that $x(\eta) = \frac{\eta}{\theta^*(\eta)}$ is increasing in η. Differentiating with respect to η we obtain

$$x_\eta(\eta) = \frac{\theta^*(\eta) - \theta^*_\eta(\eta)\,\eta}{(\theta^*(\eta))^2} > 0.$$

In other words, higher η consumers consume higher amounts of the good, and are therefore less constrained than lower η consumers.

Finally, we take these two results together to show that the net indirect utility gain of type (θ, η) is higher than the net indirect utility gain of type $(\theta, \hat{\eta})$ for $\hat{\eta} < \eta$. Recall that

$$u(\theta; \eta) = \int_{\underline{\theta}}^{\theta} x^*(z; \eta)\,dz.$$

Steps one and two have established that $x^*(\theta; \eta) \geq x^*(\theta; \hat{\eta})$ for $\hat{\eta} < \eta$, where the inequality is strict for some θ if consumers with budget $\hat{\eta}$ are constrained. It follows that $u(\theta; \eta) \geq u(\theta; \hat{\eta})$ for $\hat{\eta} < \eta$. Thus, we have shown the following:

Proposition 8 *Suppose that* $\frac{\partial}{\partial\theta}\frac{1-F(\theta|\eta)}{f(\theta|\eta)} \leq 0$ *and that* $(-\eta, \theta)$ *are affiliated and define* $\theta^*(\eta)$ *as the unique solution to*

$$\theta^*(\eta) = \frac{1 - F(\theta^*(\eta)|\,\eta)}{f(\theta^*(\eta)|\,\eta)}.$$

Then, the optimal selling procedure when both θ *and* η *are the buyer's private information is*

$$x(\theta, \eta) = \begin{cases} 0 & \text{for } \theta < \theta^*(\eta) \\ x(\eta) = \min\left\{\frac{\eta}{\theta^*(\eta)}, 1\right\} & \text{for } \theta \geq \theta^*(\eta) \end{cases}$$

with an associated payment schedule

$$t(\theta, \eta) = \begin{cases} 0 & \text{for } \theta < \theta^*(\eta) \\ \theta^*(\eta)\,x(\eta) & \text{for } \theta \geq \theta^*(\eta) \end{cases}$$

The intuition is quite simple. If the seller can force the buyer to place a bond, then exaggerating one's budget is not a feasible deviation for the buyers and only downward deviations have to be ruled out. The pointwise optimal policy (that is, the optimal policy for each η) is a take-it-or-leave-it offer at a price $\theta^*(\eta)$. So, it must not be the case that the buyer can obtain a better price by claiming his budget was lower. Hence, $\theta^*(\eta)$ must be non-increasing in η. Setting a policy with that property is indeed optimal if the standard monopoly trade-off between raising revenue per unit sold and decreasing units sold changes the right way as η is increased. When $(\theta, -\eta)$ are affiliated then the conditional distribution of θ given η has more and more mass towards the lower realizations of θ when η is increased. Hence it becomes optimal to lower the price when η increases. Since low budget types cannot mimic high budget types, the seller's policy of selling is in fact also incentive compatible.

5 Concluding Remarks

For screening problems with two unknown characteristics we have demonstrated a procedure that makes a particular subset of these problems readily solvable. The restrictions are twofold. One of the characteristics can only be mimicked in one direction and the bound on this characteristic must be part of the optimum. Secondly, the two characteristics must satisfy an affiliation property which can vary from problem to problem.

We demonstrate the effectiveness of this procedure in solving in some detail a procurement model where quantity and quality of a commodity are unknown but where agents can only mimic agent only capable of producing a lower quality. In addition, the upper bound on each agent's quality must be desired at the optimum. The procedure lets us first solve the one-dimensional screening problems conditional on the quality variable and then demonstrates that no one wishes to mimic a lower quality so that the solutions to the all of the one-dimenional problems are in fact the solutions to the overall problem. We then show briefly that a standard regulation problem and a problem proposed by Che and Gale [2000] where the seller faces possibly budget-constrained buys can be solved using this procedure.

Our approach extends to any problem that exhibits the two features we mentioned above, and indeed the logic of the argument goes beyond that. Malakhov and Vohra [2005a,2005b] and Iyengar and Kumar [2006] have studied auction problems where bidders' valuations and capacities for consumption are unknown. They show that the solution to the problem when only valuations are private information remains incentive compatible when the second dimension of private information is added. Moreover, as ours does, their result extends to any problem with similar features, that is, any problem where the principal has two choice variables, and one of them - the quantity allocated to an agent

- interacts non-trivially with the agent's types. In contrast, our results apply to problems where the principal has three choices to make and two of them - the quantity and the quality allocated to the agent - interact non-trivially with the agent's types. Taken together, these results demonstrate the usefulness of the model structures to obtain insights into the problem of multi-dimensional screening.

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