Strategic information transmission and stochastic orders

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Abstract

I develop new results on uniqueness and comparative statics of equilibria in the Crawford and Sobel (1982) strategic information transmission game. For a class of utility functions, I demonstrate that logconcavity of the density implies uniqueness of equilibria inducing a given number of Receiver actions. I provide comparative statics results with respect to the distribution of types for distributions that are comparable in the likelihood ratio order, implying, e.g., that advice from a better informed Sender induces the Receiver to choose actions that are more spread out.

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1 Introduction

Crawford and Sobel (1982) (CS henceforth) study a game of strategic information transmission between a Sender and a Receiver. The Receiver needs to take some action but is poorly informed about the state of the world. The Sender is perfectly informed about the state of the world but his ideal choice in each state differs from the Receiver's preferred choice. The Receiver asks the Sender for advice and then takes whichever action he likes best after having heard the Sender's advice. CS show that all equilibria of this game are essentially equivalent to interval partition equilibria, where the Sender divides the type space into subintervals and all Sender types within the same subinterval pool on the same message, thereby inducing the same Receiver action.

This paper provides new conditions for uniqueness of equilibria inducing a given number of Receiver actions and comparative statics results with respect to changes in the distribution of types. Both questions are intimately tied together. The known sufficient conditions for uniqueness are joint restrictions on preferences and information. As I explain in detail in section 2.3 below, this is fine as long as the distribution is kept constant. However, when the distribution is varied - as I do in the comparative statics part of the paper - then this approach becomes problematic, as the comparative statics exercise itself may expand the equilibrium set. Therefore, it is useful to have separate conditions on the utility function and on the distribution of types that jointly ensure uniqueness of equilibria inducing a given number of actions. I show that uniqueness of equilibria is guaranteed when the Receiver has preferences that make his optimal choice respond less than or at most one for one with increases in the state and the distribution has a logconcave density. The preferences form a natural class containing quadratic loss functions. Moreover, many well known distributions, such as the uniform, the (truncated) normal, the (truncated) logistic, the (truncated) extreme-value distribution, the (truncated) chi-square distribution and many more have logconcave densities.[1]

[1]Logconcave probabilities have been studied in the economics literature by An (1998) and Bagnoli and Bergström (2005). See Bagnoli and Bergström (2005) for the statement that the distributions listed above have logconcave densities, for a more extensive list of distributions with a logconcave density, and for a proof that logconcavity is preserved under truncation. It is also worth noting that logconcavity of the density implies that hazard rates are monotone nondecreasing, a property that has found widespread uses

The intuition for the uniqueness result is quite simple. Suppose the Receiver does not know the state precisely, but merely knows that the state is in some interval. If the Receiver's preferences are such that he reacts (weakly) less than one for one to increases in the state if he knows the state precisely, then, if the density is logconcave and the Receiver knows only that the state is in some interval, the Receiver reacts (weakly) less than one for one to a shifting of the interval. The reason is that for a logconcave density, the Receiver becomes relatively more pessimistic about high realizations of the state in a given interval the higher the location of the interval. I use this insight to prove uniqueness via a standard contraction mapping argument of a suitably defined composed best reply map.

Given uniqueness of equilibria for all densities in the logconcave class, one can perform comparative statics exercises with respect to changes in the distribution within a well studied class. Since the equilibrium partition is not known a priori, clear-cut comparative statics results require that the distributions can be ranked on arbitrary partitions. Building on the property that the monotone likelihood ratio property is preserved under arbitrary truncations, I show that equilibria are higher - in the sense that all Receiver actions and all marginal Sender types are higher - in a communication game with an upwardly biased Sender if the distribution is higher in the likelihood ratio order (see Shaked and Shanthikumar (2007) on stochastic orders). In a symmetric game where the Sender's and the Receiver's ideal choice agree at the prior mean and the Sender's ideal choice reacts faster to changes in the state than the Receiver's ideal choice does, I show that equilibria are more spread out - in the sense that the Receiver actions and the marginal Sender types are farther away from the prior mean - if the distribution is more spread out in the sense of a mean reverting monotone likelihood ratio property.

The latter result has direct consequences for the impact of better information on equilibrium communication. In the special case where Sender and Receiver preferences are quadratic, both players' optimal actions depend only on the conditional expectation of the state conditional on the Sender's information. Thus, better information impacts on equilibrium communication through its effects on the ex ante distribution of the conditional expectation. I show that the stochastic order needed to obtain a spreading of equilibria is consistent with notions of better information in the literature, such as riskiness of the poste-

in incentive models of contracting and mechanism design.

rior as in Blackwell (1951), a mean preserving spread in the distribution of the conditional expectation as in Szalay (2009), and the convex order considered in Ganuza and Penalva (2010). Thus, advice from a better informed Sender in this sense induces the Receiver to take actions that are more spread out. For reasons of space, I only sketch this model here, leaving applications for future research. Moreover, the focus of this paper is entirely on positive aspects. The normative side of improvements in the quality of information is studied in companion work (Eső and Szalay (2012)), where we investigate, among other things, how the marginal value of information depends on the informativeness of the communication game.

The last 30 years have witnessed an extensive body of research that has extended the CS model in various directions, some of which - but by far not all - are mentioned below. Most closely related to this work is Chen et al. (2008) and Gordon (2010, 2011). Chen et al. (2008) and Gordon (2011) study refinements among equilibria that induce different numbers of Receiver actions. This paper is concerned with uniqueness of equilibria inducing the same number of Receiver actions. While the question is different, some arguments in the refinements literature rely on uniqueness of equilibria of fixed cardinality. Therefore, this paper provides alternative sufficient conditions for these arguments as well.[2] Gordon (2010) develops a general fixed point procedure to study existence and stability of equilibria. I employ different techniques, that are useful to prove uniqueness and for comparative statics purposes once it is known that an equilibrium exists, thus complementing Gordon's approach.[3]

Very little is known on comparative statics with respect to changes in the distribution of types in the context of strategic information transmission. In contemporaneous and independent work, Chen and Gordon (2012) study stochastically monotonic shifts in the distribution of types. I am not aware of any results on spreads, nor on their connection to information in the cheap talk context. However, dispersion orders have been used in other contexts to capture better information. The most general discussion of stochastic orders on the distribution of conditional expectations is given by Ganuza and Penalva (2010); they apply their results to study an auctioneer's incentive to provide information to bidders. Szalay (2009)

[2] E.g., Proposition 3 in Chen et al. (2008) relies on uniqueness of equilibria of fixed cardinality. I thank Joel Sobel for pointing this out to me. Likewise, my uniqueness conditions imply stability of equilibria in the classical IO-sense, which is useful for some arguments used in Gordon (2011).

[3] See also Bognar et al (2008) for an analysis of a dynamic conversation game with logconcave distributions.

provides conditions on the primitives of the updating process such that better information corresponds to a more risky distribution of the conditional expectation. Inderst and Ottaviani (2011) study an application in the context of advice; an intermediary advises customers on which one of two products suits their preferences better. However, since there are only two actions, the questions they address are quite different from the ones addressed here. Szalay (2005) and more recently Xie (2012) study models of advice with information acquisition. While these papers assume that a Principal can commit on a course of action as a function of the advice he receives, this paper studies the no commitment case as in Crawford and Sobel (1982). Formalizing better information by more risky distributions allows us to analyze information acquisition in the cheap talk game in a general way.[4]

Any attempt to review the literature on strategic information transmission is bound to leave out many interesting and important contributions. Sobel (2010) provides a recent survey of the literature, that is much more complete than the following short paragraph. To name just a few contributions, in roughly chronological order, Battaglini (2002), Ambrus and Takahashi (2008), Levy and Razin (2007), and most recently Chakraborty and Harbaugh (2010) study models of multidimensional cheap talk; Dessein (2002) compares communication to delegation; Alonso et al. (2008) inquire when coordination requires centralization; Krishna and Morgan (2004) study communication allowing for multiple rounds of communication; Ottaviani and Sorensen (2006) study communication by an expert who wishes to appear well informed; Blume et al. (2007) study noisy talk; Kartik et al. (2007) and Kartik (2009) introduce costs of lying, making talk no longer cheap; Goltsmann et al. (2009) study mediated talk; Ivanov (2012) studies informational control by the Receiver. Many exciting questions involve changes in the distribution of types, that have -to the best of my knowledge- not been addressed so far. So, this paper hopefully proves useful to address such questions in future research.

The remainder of the paper is structured as follows: in section two, I introduce the model alongside with known results about it and explain in more detail why separate conditions on preferences and information that jointly ensure uniqueness are useful; section three discusses

[4] An alternative route is to study a specific statistical model. Specific statistical models of information acquisition in the cheap talk game are studied in Argenziano et al. (2011) (beta-binomial experiments), Eső and Szalay (2012) (all-or-nothing information acquisition), and Arean and Szalay (2012) (normally distributed signals and priors).

impacts of changes in the stochastic structure on the Receiver's optimal choice; section four demonstrates the uniqueness result; section five provides some comparative statics results with respect to the distribution of types. Section six derives a statistical model that allows to capture the notion of better information in the game of strategic information transmission in a useful and general way. The final section concludes. All proofs are gathered in the appendix.

2 The model

2.1 Setup

I analyze the strategic information transmission game by Crawford and Sobel (1982). There are two players, a Sender and a Receiver. The Receiver needs to take an action, y. The Receiver is uncertain about a state of the world, ω, that influences the ideal action he would like to take. The Receiver knows only that the state ω is drawn from a distribution with continuously differentiable cdf $F(\omega)$ and density $f(\omega) > 0$ on the support $[0,1]$. Prior to taking the action, the Receiver gets advice from a Sender who knows ω; that is, the Sender sends a message $m \in M$ to the Receiver, where M is a rich message space. After the Receiver has heard the Sender's advice, the Receiver takes whichever action he finds optimal at that point; thus, there is no ex ante commitment to a course of action as a function of what the Sender says. Finally, payoffs are realized and the game ends. The information structure is common knowledge.

The Players' utility functions depend on the action y and the state of the world ω. The Sender's utility $U^S(y,\omega)$ and the Receiver's utility $U^R(y,\omega)$ satisfy the following assumptions: for each ω, there exists y such that $U_1^j(y,\omega) = 0$ for $j = R, S$; moreover, $U_{11}^j(y,\omega) < 0$ and $U_{12}^j(y,\omega) > 0$ for $j = R, S$. Subscripts denote partial derivatives. Hence, for each ω, each player has a unique ideal choice $y^j(\omega)$ and this ideal choice is differentiable and strictly increasing in ω, as $\frac{dy^j(\omega)}{d\omega} = -\frac{U_{12}^j(y^j(\omega),\omega)}{U_{11}^j(y^j(\omega),\omega)}$. For future reference, note that $\frac{dy^j(\omega)}{d\omega} \leq (\geq) 1$ for all ω if $U_1^j(y,\omega) + U_2^j(y,\omega)$ is nonincreasing (nondecreasing) in y. Likewise for future reference, define the difference between the ideal choices of Sender and Receiver, that is the bias, as

$$b(\omega) \equiv y^S(\omega) - y^R(\omega).$$

Notice that $b(\omega)$ is differentiable and allowed to depend on the state ω.

2.2 Known results

CS show that any Bayesian equilibrium of this game is essentially equivalent to an interval partition of the unit interval, where Sender types within the same partition element pool on the same message. For $b(\omega) > 0$ for all ω, the Receiver takes a finite number of distinct actions in equilibrium.

Interval partitional equilibria are described by a partition of $[0,1]$ into N non-degenerate intervals, defined by the thresholds $(a_0^N, \ldots, a_N^N)$ such that $0 = a_0^N < a_1^N < \ldots < a_N^N = 1$. There are N different messages needed to sustain the equilibrium, m_i for $i = 1, ..., N$. Let $P_i^N = \left[a_{i-1}^N, a_i^N\right]$ denote the i^{th} element of a partition with N elements. Types $\omega \in P_i^N$ send message m_i and the Receiver's best response to message m_i is to pick the action

$$y_i = y_i\left(a_{i-1}^N, a_i^N\right) \equiv \arg\max_y \int_{a_{i-1}^N}^{a_i^N} U^R(y,\omega) f(\omega)\, d\omega \tag{1}$$

The overall construction is an equilibrium if indeed all types $\omega \in [a_{i-1}^N, a_i^N]$ weakly prefer to send message m_i rather than any other message m_j. Given the assumed preferences, the most tempting deviations are to mimic types in adjacent partition elements. Using (1), the indifference condition for type a_i^N reporting either $\omega \in P_i^N$ (that is, sending message m_i) or $\omega \in P_{i+1}^N$ (message m_{i+1}) is

$$U^S\left(y_i, a_i^N\right) = U^S\left(y_{i+1}, a_i^N\right). \tag{2}$$

With y_i as defined in (1), (2) forms a system of $N-1$ equations; initial and final condition are $a_0^N = 0$ and $a_N^N = 1$, respectively. CS prove that for each given, and strictly positive divergence of interests between the Sender and the Receiver, measured by the function $b(\cdot)$, there is an integer $N(b(\cdot))$ such that (2) has at least one solution for $N = \{1, \ldots, N(b(\cdot))\}$ [5]. Gordon (2010) demonstrates the existence of infinite equilibria for the case where $b(0) < 0 < b(1)$. For convenience, I state these results in the following Lemma:

[5] Note a slight departure of CS's notation; in CS, b is a parameter that measures the closeness of preferences; here the function $b(\cdot)$ is taken to measure the difference in ideal choices directly. However, this difference is purely notational.

Lemma 1 *(Crawford and Sobel (1982)) For each $b(\cdot)$, where $b(\omega) > 0$ for all ω, there is $N(b(\cdot))$ such that (2) has at least one solution for $N = \{1, \ldots, N(b(\cdot))\}$.*
(Gordon (2010)) If $b(0) < 0 < b(1)$, then (2) has a solution for any N.

For the proofs of these statements, see Crawford and Sobel (1982) and Gordon (2010), respectively. Consistently with these results, I impose the following:

Assumption: $b(\omega)$ is non-decreasing.

CS show that if the solutions to the system of equations (2) satisfy a monotonicity condition (M), then there is only one solution for a given $N \leq N(b(\cdot))$. Formally, condition (M) requires that if $(\tilde{a}_0^N, \ldots, \tilde{a}_N^N)$ and $(\hat{a}_0^N, \ldots, \hat{a}_N^N)$ are two solutions of (2) with $\tilde{a}_0^N = \hat{a}_0^N$ and $\hat{a}_1^N > \tilde{a}_1^N$, then $\hat{a}_i^N > \tilde{a}_i^N$ for $i \geq 2$. The known sufficient conditions for condition M (see Theorem 2 in CS) are

$$U_1^S(y,\omega) + U_2^S(y,\omega) \text{ is nondecreasing in } y \tag{3}$$

and

$$\int_0^a U_{11}^R(y,\omega) f(\omega)\, d\omega + U_1^R(y,a) f(a) \text{ is nonincreasing in } a. \tag{4}$$

2.3 The Agenda

Conditions (3) and (4) are joint restrictions on the Receiver's utility function and the distribution of types. This is fine as long as the distribution is constant. To see this, recall from CS[6] that (3) and (4) hold if the distribution is uniform and there are concave functions $\overline{U}^j$, for $j = R, S$, such that $U^R(y,\omega) = \overline{U}^R(y-\omega)$ and $U^S(y,\omega) = \overline{U}^S(y-\omega, b)$, where b does not depend on ω. If the distribution is nonuniform, then we can take $\hat{\omega} \equiv F(\omega)$ as the state of the world and rewrite preferences using $\omega \equiv F^{-1}(\hat{\omega})$. If the rescaled preferences, with $\hat{\omega}$ as the state of the world, satisfy conditions (3) and (4), then uniqueness of equilibria is guaranteed. An example in this spirit is when there are concave functions $\overline{U}^R$ and $\overline{U}^S$, respectively, such that $U^R(y,\omega) = \overline{U}^R(y - F(\omega)) = \overline{U}^R(y - \hat{\omega})$ and

[6] See their remark after their Theorem 2.

$U^S(y,\omega) = \overline{U}^S(y - F(\omega), b) = \overline{U}^S(y - \hat{\omega}, b)$. This argument works fine as long as the distribution is constant. However, if the object of research is a comparative statics exercise that changes the distribution of types, then the comparative statics exercise itself affects whether conditions (3) and (4) are satisfied. If G is an alternative distribution, then the state needs to be rescaled with respect to the distribution G. The same preferences as a function of the rescaled state of the world would now be written as $U^R(y,\omega) = \overline{U}^R(y - F(G^{-1}(\hat{\omega})))$, so the relevant derivative in condition (4) would depend on $1 - \frac{f}{g}$, a factor that necessarily changes sign over its domain. Thus, changing the distribution from F to G may expand the equilibrium set.

So, if we wish to engage in comparative statics exercises involving changes in the distribution of types and wish to ensure that equilibria are unique for all distributions that we allow for, then we need separate conditions on the preferences and the distribution of types that jointly ensure uniqueness of equilibria given N. I prove the uniqueness of equilibria of given size N if the Sender utility function satisfies condition (3), the Receiver utility function satisfies

$$U_1^R(y,\omega) + U_2^R(y,\omega) \text{ is nonincreasing in } y \tag{5}$$

and

$$f(\omega) \text{ is logconcave.} \tag{6}$$

Condition (5) describes a natural class of preferences for which the Receiver's optimal choice responds less than one for one to increases in the state. Condition (6) describes a natural class of distributions, as explained in the introduction. Conditions (5) and (6) are neither stronger nor weaker than condition (4); neither do conditions (5) and (6) imply condition (4) nor is the reverse true. The rescaling argument of CS can be applied also to densities that are not logconcave; likewise, if we take preferences $U^R(y,\omega) = \overline{U}^R(y-\omega)$ and $U^S(y,\omega) = \overline{U}^S(y-\omega, b)$, then (5) and (6) are satisfied for the whole class of logconcave densities, while (4) holds only for the uniform distribution. For the purpose of comparative statics with respect to the distribution of types conditions (5) and (6) have the advantage that varying the distribution within the class of logconcave densities can never lead to multiplicity of equilibria for given N.

Before proving uniqueness of equilibria and engaging in comparative statics, we need to understand how the Receiver's best reply reacts to changes in the information structure.

3 The Receiver's best reply and the information structure

Consider the Receiver's decision problem. If he knows the state of the world, ω, then he chooses in each state of the world the act $y^R(\omega)$, defined by the condition

$$U_1^R\left(y^R(\omega),\omega\right)=0.$$

Consider now the Receiver's decision problem, when he does not know the state of the world precisely, but knows that $\omega\in[\underline{x}+\Delta,\overline{x}+\Delta]$, where $0\leq\underline{x}+\Delta<\overline{x}+\Delta<1$. In that case the solution to the Receiver's problem becomes

$$y(\underline{x}+\Delta,\overline{x}+\Delta;h)=\arg\max_y\int_{\underline{x}+\Delta}^{\overline{x}+\Delta}U^R(y,\omega)\,h(\omega)\,d\omega. \tag{7}$$

where $h\in f,g$ denote two alternative densities. $\underline{x}$, $\overline{x}$, Δ, and h describe elements of the Receiver's information structure. How does his decision depend on these objects? We are interested in describing the class of densities for which it is true that $y(\underline{x},\overline{x};g)>y(\underline{x},\overline{x};f)$ for arbitrary truncations $\underline{x},\overline{x}$.[7] Moreover, we wish to know for which classes of densities it is true that $\frac{\partial}{\partial\Delta}y(\underline{x}+\Delta,\overline{x}+\Delta)\leq 1$ for a given density f.

Lemma 2 *For any $\underline{x}$ and $\overline{x}$ such that $0\leq\underline{x}<\overline{x}\leq 1$,*

$$y(\underline{x},\overline{x};g)>y(\underline{x},\overline{x};f)$$

if $\frac{g}{f}$ is strictly increasing in ω for all $\omega\in[0,1]$.
Conversely, if there is $\underline{x}$ and $\overline{x}$ such that $0\leq\underline{x}<\overline{x}\leq 1$ and $\frac{g}{f}$ is nonincreasing in ω for $\omega\in[\underline{x},\overline{x}]$ then $y(\underline{x},\overline{x};g)\leq y(\underline{x},\overline{x};f)$.

If $\frac{g}{f}$ is strictly increasing in ω for all $\omega\in[0,1]$ then g is said to be higher in the likelihood ratio order than f (see Shaked and Shanthikumar (2007)). Thus, the Receiver believes that the high realizations of the state are relatively more likely when the state of the world

[7] Since Δ is constant in this exercise and the truncation is arbitrary, it is without loss of generality to set $\Delta=0$ to characterize this class of densities.

follows the distribution with density g than when it follows the distribution with density f. Since $U_{12}^R > 0$, the Receiver responds to the change in the density from f to g by increasing his optimal choice. As a partial converse, if there is a subinterval over which $\frac{g}{f}$ is strictly decreasing, then the inequality in the lemma would be reversed.

Consider now the second question:

Lemma 3 *$y(\underline{x}, \overline{x}) = \arg\max_y \int_{\underline{x}}^{\overline{x}} U^R(y, \omega) f(\omega) d\omega$ satisfies for any $\underline{x}$ and $\overline{x}$ such that $0 \leq \underline{x} < \overline{x} < 1$ and for any Receiver utility function such that $U_1^R(y, \omega) + U_2^R(y, \omega)$ is nonincreasing in y*

$$\frac{\partial}{\partial \underline{x}} y(\underline{x}, \overline{x}) + \frac{\partial}{\partial \overline{x}} y(\underline{x}, \overline{x}) \leq 1 \tag{8}$$

if $f(\omega)$ is logconcave.
Conversely, if there is there is an interval $[\underline{\omega}, \overline{\omega}]$ such that $f(\omega)$ is logconvex for all $\omega \in [\underline{\omega}, \overline{\omega}]$, then there is a utility function with $U_1^R(y, \omega) + U_2^R(y, \omega)$ nonincreasing in y such that $\frac{\partial}{\partial \underline{x}} y(\underline{x}, \overline{x}) + \frac{\partial}{\partial \overline{x}} y(\underline{x}, \overline{x}) \geq 1$ for $\underline{\omega} \leq \underline{x} < \overline{x} < \overline{\omega}$.

To understand this result, it proves useful to compare decision problem (7) with a decision problem where where $h = f$ and $\Delta \equiv 0$, which has solution

$$y(\underline{x}, \overline{x}) = \arg\max_y \int_{\underline{x}}^{\overline{x}} U^R(y, \omega) f(\omega) d\omega. \tag{9}$$

The decision problems (9) and (7) differ in that the support of the random variable in the former problem is higher than in the latter problem. Changing variables to $\tilde{\omega} = \omega - \Delta$ in problem (7) makes the supports of both problems equal to $[\underline{x}, \overline{x}]$ and changes the integrand in the latter problem to $U^R(y, \omega + \Delta) f(\omega + \Delta)$. So, there are two effects when Δ is increased: a direct effect on the utility function and a second effect that the density is changed from $f(\omega)$ to $f(\omega + \Delta)$. Consider first the direct effect for a constant density. By $U_{12}^R > 0$, the Receiver wishes to increase his optimal action when the state is higher. Since $U_1^R(y, \omega) + U_2^R(y, \omega)$ is nonincreasing in y, he wishes to increase his optimal action less than one for one with increases in the state. The second effect turns out to be just another incarnation of the preceding lemma. To see this, recall from An (1998) that $f(\omega)$ is logconcave if and only if

$$f(\omega' + \theta) f(\omega'') \geq f(\omega'' + \theta) f(\omega')$$

for all ω', ω'' and θ such that $0 \leq \omega' < \omega'' \leq \omega'' + \theta \leq 1$. Since I assume that the distribution has full support, this is equivalent to

$$\frac{f(\omega' + \theta)}{f(\omega'' + \theta)} \geq \frac{f(\omega')}{f(\omega'')}.$$

Taking $\Delta \equiv \omega'' - \omega' > 0$, this is equivalent to

$$\frac{f(\omega)}{f(\omega + \Delta)} \text{ nondecreasing in } \omega. \tag{10}$$

(10) is an upshifted likelihood ratio order.[8] Thus, when the decision problem is changed from (9) to (7), then the Receiver becomes relatively more pessimistic about the state of the world, in the sense that he now believes high outcomes are relatively less likely than before in the sense of the monotone likelihood ratio property, (10). Therefore, the effect on the Receiver's optimal choice $y(\underline{x} + \Delta, \overline{x} + \Delta)$, arising through the changed inference about the state of the world, reinforces the effect through the utility function where increases in the state induce a less than one for one reaction in the Receiver's optimal choice.

Taking the Receiver's utility as a quadratic function isolates the statistical effects on the Receiver's decision problem. For this utility function, we have $y(\underline{x} + \Delta, \overline{x} + \Delta) = \mathbb{E}[\omega | \underline{x} + \Delta \leq \omega \leq \overline{x} + \Delta]$ and the proof of the lemma specializes then to show that

$$\mathbb{E}[\omega | \underline{x} + \Delta \leq \omega \leq \overline{x} + \Delta] \leq \mathbb{E}[\omega | \underline{x} \leq \omega \leq \overline{x}] + \Delta.$$

Rearranging, dividing by Δ, and taking limits as Δ goes to zero, we have the following Corollary:

Corollary 1 *For any $\underline{x}$ and $\overline{x}$ such that $0 \leq \underline{x} < \overline{x} \leq 1$,*

$$\frac{\partial}{\partial \underline{x}} \mathbb{E}[\omega | \underline{x} \leq \omega \leq \overline{x}] + \frac{\partial}{\partial \overline{x}} \mathbb{E}[\omega | \underline{x} \leq \omega \leq \overline{x}] \leq (\geq) 1 \tag{11}$$

if $f(\omega)$ is logconcave (logconvex).

[8] See Shaked and Shanthikumar (2007). A proof of the equivalence between the upshifted likelihood ratio order and logconcavity appears also in Lillo et. al (2001).

To summarize, the optimal choice $y(\underline{x}, \overline{x})$ is increased by a change in the density for arbitrary truncations if the densities satisfy the monotone likelihood ratio property. For logconcave densities, $y(\underline{x}, \overline{x})$ depends on the bounds of truncation as follows: by the full support assumption and the fact that $U_{12}^R > 0$, I have $\frac{\partial}{\partial \underline{x}} y(\underline{x}, \overline{x})$ and $\frac{\partial}{\partial \overline{x}} y(\underline{x}, \overline{x}) > 0$. By the lemma, I have $\frac{\partial}{\partial \underline{x}} y(\underline{x}, \overline{x}) + \frac{\partial}{\partial \overline{x}} y(\underline{x}, \overline{x}) \leq 1$. Together, these results imply also (the known result) that $\frac{\partial}{\partial \underline{x}} y(\underline{x}, \overline{x}) < 1$ and $\frac{\partial}{\partial \overline{x}} y(\underline{x}, \overline{x}) < 1$.[9]

4 Uniqueness

Suppose from now on that conditions (3), (5) and (6) hold. These assumptions imply uniqueness of the solution to the system (2) for given N, whenever a solution with N partition elements exists. The most direct way to see this, is to observe that these conditions imply condition (M) in CS[10]. However, I find the following arguments more illuminating.

To illustrate the idea in simple terms, consider a three-partition equilibrium, which satisfies $a_0^3 = 0$, $a_3^3 = 1$, and a_1^3,a_2^3 are determined by

$$\Phi_1\left(a_1^3, a_2^3\right) \equiv U^S\left(y_1\left(0, a_1^3\right), a_1^3\right) - U^S\left(y_2\left(a_1^3, a_2^3\right), a_1^3\right) = 0 \tag{12}$$

and

$$\Phi_2\left(a_2^3, a_1^3\right) \equiv U^S\left(y_2\left(a_1^3, a_2^3\right), a_2^3\right) - U^S\left(y_3\left(a_2^3, 1\right), a_2^3\right) = 0. \tag{13}$$

For given a_2^3, equation (12) determines a_1^3; let $\tilde{\Phi}_1\left(a_2^3\right)$ denote the solution of (12) for given a_2^3. Likewise, equation (13) determines a_2^3 as a function of a_1^3; let $\tilde{\Phi}_2\left(a_1^3\right)$ denote the solution of (13) for given a_1^3. We can think of these functions as of best reply functions in a simultaneous

[9] While it is well known that $\frac{\partial}{\partial \underline{x}} y(\underline{x}, \overline{x}) < 1$ and $\frac{\partial}{\partial \overline{x}} y(\underline{x}, \overline{x}) < 1$ for logconcave densities, to the best of my knowledge, the equivalence between logconcavity and condition (8) has not been noted in the literature. To be sure, not even the simpler condition (11) has been noted. Logconcave densities have been used extensively in reliability theory, where typical questions concern the residual lifetime of an object. An (1998) and Bagnoli and Bergstrom (2005) prove the connections between logconcave probability and various other conditions that are frequently used in economic analysis. Reliability theory uses one-sided measures and studies, e.g., how a failure rate changes as an object ages. Formally, this amounts to changes in one truncation point in a truncated distribution. In the present context, it is useful to analyze a simultaneous moving of both points of truncation.

[10] A proof of this result is available from the author upon request.

move game, where a "high" player chooses $a_2^3 \in [a_1^3, 1]$ and a "low" player chooses $a_1^3 \in [0, a_2^3]$. Obviously, there is no such game being played, but the point is that the situation can be analyzed as if such a game were played. An equilibrium must satisfy the following fixed point condition

$$a_1^3 = \tilde{\Phi}_1\left(\tilde{\Phi}_2\left(a_1^3\right)\right). \tag{14}$$

Consider the slope of the composed "best reply" on the right-hand side, $\tilde{\Phi}_1'\left(\tilde{\Phi}_2\left(a_1^3\right)\right)\tilde{\Phi}_2'\left(a_1^3\right)$. By the implicit function theorem, I have $\tilde{\Phi}_1'\left(a_2^3\right) = \frac{da_1^3}{da_2^3} = \frac{\frac{\partial}{\partial a_2^3}\Phi_1\left(a_1^3,a_2^3\right)}{-\frac{\partial}{\partial a_1^3}\Phi_1\left(a_1^3,a_2^3\right)}$ and so[11]

$$\frac{da_1^3}{da_2^3} = \frac{U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)\frac{\partial y_2}{\partial a_2^3}}{U_1^S\left(y_1\left(0,a_1^3\right),a_1^3\right)\frac{\partial y_1}{\partial a_1^3} + U_2^S\left(y_1\left(0,a_1^3\right),a_1^3\right) - U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)\frac{\partial y_2}{\partial a_1^3} - U_2^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)} \tag{15}$$

Likewise, $\tilde{\Phi}_2'\left(a_1^3\right) = \frac{da_2^3}{da_1^3} = \frac{\frac{\partial}{\partial a_1^3}\Phi_2\left(a_2^3,a_1^3\right)}{-\frac{\partial}{\partial a_2^3}\Phi_2\left(a_2^3,a_1^3\right)}$ and so

$$\frac{da_2^3}{da_1^3} = \frac{U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_2^3\right)\frac{\partial y_2}{\partial a_1^3}}{U_1^S\left(y_2\left(a_2^3,1\right),a_2^3\right)\frac{\partial y_2}{\partial a_2^3} + U_2^S\left(y_2\left(a_2^3,1\right),a_2^3\right) - U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_2^3\right)\frac{\partial y_2}{\partial a_2^3} - U_2^S\left(y_2\left(a_1^3,a_2^3\right),a_2^3\right)}. \tag{16}$$

Conditions (3), (5) and (6) imply that $\tilde{\Phi}_1'\left(a_2^3\right) \in (0,1)$ and $\tilde{\Phi}_2'\left(a_1^3\right) \in (0,1)$. Since the proof of the Theorem below discusses the arguments at length, I merely sketch the arguments here. Consider (15) first. The indifference condition of the Sender, $U^S\left(y_1\left(0,a_1^3\right),a_1^3\right) = U^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)$, and the fact that $U_{12}^S > 0$, imply that $y_1\left(0,a_1^3\right) < y^S\left(a_1^3\right) < y_2\left(a_1^3,a_2^3\right)$. Hence, by $U_{11}^S < 0$, I have $U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right) < 0$ and thus that the numerator on the right-hand side of (15) is positive. From conditions (5) and (6) I have $\frac{\partial y_1\left(0,a_1^3\right)}{\partial a_1^3}, \frac{\partial y_2\left(a_1^3,a_2^3\right)}{\partial a_1^3} < 1$. Together with condition (3), this implies that the denominator is negative as well. Using these properties, I have $\tilde{\Phi}_1'\left(a_2^3\right) < 1$ iff

$$\begin{aligned} & U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)\left(\frac{\partial y_2\left(a_1^3,a_2^3\right)}{\partial a_1^3} + \frac{\partial y_2\left(a_1^3,a_2^3\right)}{\partial a_2^3}\right) + U_2^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right) \\ > \; & U_1^S\left(y_1\left(0,a_1^3\right),a_1^3\right)\frac{\partial y_1\left(0,a_1^3\right)}{\partial a_1^3} + U_2^S\left(y_1\left(0,a_1^3\right),a_1^3\right). \end{aligned} \tag{17}$$

[11] The dependence of $\frac{\partial y_i}{\partial a_j^3}$ for $i,j = 1,2$ is suppressed for reasons of space.

Again using the indifference condition of the Sender, I have $U_1^S\left(y_1\left(0,a_1^3\right),a_1^3\right) > 0 > U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)$. Together with the characterization of logconcave densities from the Lemma, (8), it is now easy to see that the left-hand side of this inequality is weakly larger than $U_1^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)+U_2^S\left(y_2\left(a_1^3,a_2^3\right),a_1^3\right)$ while the right-hand side is strictly smaller than $U_1^S\left(y_1\left(0,a_1^3\right),a_1^3\right)+U_2^S\left(y_1\left(0,a_1^3\right),a_1^3\right)$. Obviously, condition (3) then implies that (17) is indeed satisfied.

Virtually the same argument can be made to show that conditions (3), (5) and (6) imply that $\tilde{\Phi}_2'\left(a_1^3\right) < 1$, the only difference being that now both the numerator and the denominator in (16) are positive. Hence, logconcavity of the density implies, for the class of Sender and Receiver utility functions considered, that the composed function $\tilde{\Phi}_1\left(\tilde{\Phi}_2\left(a_1^3\right)\right)$ is a contraction mapping. This insight applies equally well to equilibria with an arbitrary number of partition elements. I have the following Theorem:

Theorem 1 *Suppose that (2) has a solution for given N. Suppose also that $U_1^S(y,a)+U_2^S(y,a)$ is nondecreasing in y, that $U_1^R(y,a)+U_2^R(y,a)$ is nonincreasing in y and that the distribution has a logconcave density. Then there is only one solution inducing N distinct Receiver choices.*

The formal proof in the appendix is very similar to the original one given given by CS. The key difference is the sufficient conditions. Consider the sequence $\left\{x,a_2^N(x),\ldots,a_{N-1}^N(x),1\right\}$ for given initial value $x\in[0,1)$. In contrast to the equilibrium sequence of thresholds $\left\{0,a_1^N,\ldots,a_{N-1}^N,1\right\}$, a_1^N is replaced by the value x. x is an initial condition for the sequence $\left\{x,a_2^N(x),\ldots,a_{N-1}^N(x),1\right\}$ and is allowed to take an arbitrary value. Clearly, the entire sequence depends on the initial condition x. For Sender utility functions such that $U_1^S(y,a)+U_2^S(y,a)$ is nondecreasing in y, logconcavity of the density allows me to prove that each of the threshold points $a_i^N(x)$ for $i=2,\ldots,N-1$ increases with x and does so at a rate smaller than unity. In particular, the threshold $a_2^N(x)$ satisfies $\frac{da_2^N(x)}{dx}\in(0,1)$.

An equilibrium sequence of thresholds must satisfy in addition the condition

$$U^S\left(y_1\left(a_0^N,x\right),x\right)=U^S\left(y_2\left(x,a_2^N(x)\right),x\right). \tag{18}$$

Condition (18) is again a fixed-point condition. However, $a_2^N(x)$ depends on x in a fairly complicated way. (18) together with $U_{12}^S>0$ imply that $y_1\left(a_0^N,x\right)<y^S(x)<y_2\left(x,a_2^N(x)\right)$

and therefore $U_1^S\left(y_1\left(a_0^N,x\right),x\right)>0>U_1^S\left(y_2\left(x,a_2^N\left(x\right)\right),x\right).$ Using these insights, I prove that, for Sender utility functions such that $U_1^S\left(y,a\right)+U_2^S\left(y,a\right)$ is nondecreasing in y and logconcave densities, the effect of a change in x on the left-hand side,

$$U_1^S\left(y_1\left(a_0^N,x\right),x\right)\frac{\partial y_1\left(a_0^N,x\right)}{\partial x}+U_2^S\left(y_1\left(a_0^N,x\right),x\right)$$

is always smaller than the effect of a marginal increase in x on the right-hand side,

$$U_1^S\left(y_2\left(x,a_2^N\left(x\right)\right),x\right)\left(\frac{\partial y_2\left(x,a_2^N\left(x\right)\right)}{\partial x}+\frac{\partial y_2\left(x,a_2^N\left(x\right)\right)}{\partial a_2^N\left(x\right)}\frac{da_2^N\left(x\right)}{dx}\right)+U_2^S\left(y_2\left(x,a_2^N\left(x\right)\right),x\right).$$

Hence, there is at most one solution to (18) in x.[12]

5 Comparative statics

I now turn to comparative statics with respect to changes in the distribution of types. I first consider stochastically monotonic shifts and then proceed to analyze spreads in a symmetric model.

5.1 Monotonic shifts and onesided biases

Consider an alternative distribution of the Sender's type with density $g\left(\omega\right)$ and cdf $G\left(\omega\right).$ Suppose that $g\left(\omega\right)$ is logconcave just as $f\left(\omega\right)$ is and suppose that both communication games with type distributions f and g, respectively, have an equilibrium inducing N distinct Receiver actions. Let $y_i^N\left(h\right)$ denote the Receiver's optimal choice conditional on the density being h and on $\omega\in P_i^N\left(h\right)$ for $h=f,g$ and let $a_i^N\left(h\right)$ denote the i^{th} threshold when the distribution is h.

Rewriting (2) as a system of "best-reply" functions, the unique equilibrium for given N satisfies

$$U^S\left(y_i^N\left(h\right),a_i^N\left(h\right)\right)=U^S\left(y_{i+1}^N\left(h\right),a_i^N\left(h\right)\right)\text{ for }i=1,\ldots,N-1.\tag{19}$$

[12] There is a second difference to the proof in CS, which is however not important. CS consider the sequence $\left\{0,x,a_2^N\left(x\right),\ldots,a_{N-1}^N\left(x\right),a_N^N\left(x\right)\right\}$ and show that there is a unique x such that $a_N^N\left(x\right)=1$. Obviously, both arguments are identical. The exposition given above emphasizes the connection to the fixed point and contraction mapping arguments most clearly.

$y_i^N(h)$ is an increasing function of $a_{i-1}^N(h)$ and $a_i^N(h)$; $y_{i+1}^N(h)$ is an increasing function of $a_i^N(h)$ and $a_{i+1}^N(h)$. Both functions do not directly depend on all non-adjacent thresholds $a_j^N(h)$ for $j \neq i-1, i, i+1$. As part of the proof of the Theorem below, I show that $a_i^N(h)$ is strictly increasing in $a_{i-1}^N(h)$ and $a_{i+1}^N(h)$. Thus, (19) forms a system of "best-response"-functions such that $a_i^N(h)$ is non-decreasing in $\mathbf{a}_{-i}^N$, the vector of the remaining thresholds and strictly increasing in the adjacent thresholds $a_{i-1}^N(h)$ and $a_{i+1}^N(h)$. Hence, we can build on comparative statics methods developed for games with strategic complementarities (see Vives (1990, 1999)). In particular, if the change from f to g weakly increases $y_i^N(h)$ and $y_{i+1}^N(h)$, for $i = 1, \ldots, N-1$, then the equilibrium thresholds in the game with density g are higher than in the game with density f. Recalling that Lemma 2 has established that $y_i^N(g) > y_i^N(f)$ for arbitrary truncations if g is higher than f in the likelihood ratio order, the following theorem is obvious:

Theorem 2 *Suppose that two densities f and g are both logconcave. Suppose further that the communication games when the state is distributed according to f and g, respectively, each have an equilibrium inducing N distinct Receiver actions. If $b(\omega) > 0$ and $\frac{g}{f}$ is strictly increasing in ω, then $a_i^N(g) > a_i^N(f)$ and $y_i^N(g) > y_i^N(f)$ for $i = 1, \ldots, N-1$.*

The strict monotone likelihood ratio property (MLRP) implies that the induced Receiver actions when the density is g are higher than the induced Receiver actions when the density is f for arbitrary truncations resulting from the Sender strategy. To restore the system of indifference condtions for the threshold types, (19), each threshold a_i^N needs to be increased for given adjacent thresholds a_{i-1}^N and a_{i+1}^N. Thus, changing the distribution from f to g amounts to an upward shift of the best reply functions in a system of nondecreasing best replies. Hence, we obtain strict monotone comparative statics.

5.2 Spreads

Spreads are more difficult to analyze than (stochastically) monotonic shifts in the distribution. However, the logic behind Theorem 2 extends readily to spreads in a symmetric model. To make the model symmetric, I restrict the density and preferences as follows:

The density is symmetric around its mean value, one half. The Receiver's utility is a quadratic loss function

$$U^R(y,\omega) = -r(y-\omega)^2.$$

where $r > 0$ is a constant.[13] Finally, the Sender's utility function is symmetric around its bliss point, $y^S(\omega) = \omega + b(\omega)$, so

$$U^S(y,\omega) = \overline{U}^S(y - (\omega + b(\omega))).$$

where $\overline{U}^S$ is strictly concave with a peak at zero. For this specification, the optimal choices from the Receiver's and the Sender's perspective are $y^R(\omega) = \omega$ and $y^S(\omega) = \omega + b(\omega)$, respectively, and the equilibrium condition (2) simplifies to

$$\left(a_i^N + b\left(a_i^N\right)\right) - \mathbb{E}\left[\omega|\omega \in \left[a_{i-1}^N, a_i^N\right]\right] = \mathbb{E}\left[\omega|\omega \in \left[a_i^N, a_{i+1}^N\right]\right] - \left(a_i^N + b\left(a_i^N\right)\right).$$

$b(\omega)$ satisfies $b\left(\frac{1}{2}+\delta\right) = -b\left(\frac{1}{2}-\delta\right)$ for all $\delta \in \left[0, \frac{1}{2}\right]$. Consistently with the assumptions made above, I assume that the Sender's bias is positive for values above the mean and negative for values below the mean. Obviously, I can construct a symmetric equilibrium around the mean value one half - and I do so in the appendix. By logconcavity of the distribution, for each given N, the equilibrium is unique. The following result is then immediate:

Theorem 3 *Consider a symmetric model as outlined above and two densities f and g that are both symmetric and logconcave. Suppose further that $\frac{g}{f}$ is strictly decreasing in ω for $\omega < .5$ and that $\frac{g}{f}$ is strictly increasing in ω for $\omega > .5$ and consider any equilibrium with finite N. Then, $\left|a_i^N(g) - .5\right| \geq \left|a_i^N(f) - .5\right|$ and $\left|y_i^N(g) - .5\right| \geq \left|y_i^N(f) - .5\right|$ for $i = 1, \ldots, N-1$.*

[13] It is possible to extend the model to the generalized quadratic loss function analyzed in Alonso and Matoushek (2008). As Alonso and Matouschek (2008) note, a game where the Receiver has a generalized quadratic loss function with a state dependent weight $r(\omega)$ can be analyzed as if the Receiver just had a standard quadratic loss function by merging the weight $r(\omega)$ with the density as follows: notice that $h(\omega|\omega_1,\omega_2) \equiv r(\omega) f(\omega) / \int_{\omega_1}^{\omega_2} r(t) f(t)\, dt$ is a well defined probability density function for any $0 \leq \omega_1 < \omega_2 \leq 1$. Using this adjusted density, the Receiver simply minimizes the standard quadratic loss function. Note that with a state dependent weight, the merged density needs to be logconcave.

To avoid the notational clutter, I assume that $r(\omega) \equiv r > 0$. However, it should be kept in mind, that the results carry over in straightforward manner to the case of a state dependent weight.

If N is even, then all inequalities are strict; if N is odd, then $a^N_{\frac{N+1}{2}}(f) = a^N_{\frac{N+1}{2}}(g) = .5$, and all inequalities are strict but for $i = \frac{N+1}{2}$.

The densities in the theorem satisfy a strict monotone likelihood ratio property on the half-supports. Obviously, this ordering is not preserved under arbitrary truncations but only under truncations on the half support. However, given symmetry it suffices to rank distributions on the half supports to tell how equilibria change when the distribution changes. Hence, by the same rationale as in Theorem 2, the equilibrium thresholds and the induced choices are more spread out under distribution g than under distribution f.

Obviously, symmetry is a restrictive assumption. It is well known that logconcave distributions are strongly unimodal (see Dharmadhikari and Joag-dev (1988), Theorem 1.10). Hence, symmetric, logconcave densities have their mode at the mean. However, the assumption buys a lot of tractability. In particular, better information can be modeled by more dispersed distributions. Hence, Theorem 3 is useful to study the effects of improved information in the CS model in a fairly general way. I now derive the statistical details of such a model.

6 The effects of improved information on communication

Let both the Sender's and the Receiver's utility functions be quadratic, so $U^R(y,\omega) = -(y-\omega)^2$ and $U^S(y,\omega) = -(y-(\omega+b(\omega)))^2$, where $b(\omega) = \omega - \mathbb{E}_\omega[\omega]$ for the sake of concreteness[14]. Suppose that neither the Sender nor the Receiver know the state of the world, ω; however, the Sender obtains a signal, s, about the state of the world. Conditional on s, the optimal choices of the Receiver and the Sender, respectively, are $y^R(s) = \mathbb{E}_\omega[\omega|\, s]$ and $y^S(s) = 2\mathbb{E}_\omega[\omega|\, s] - \mathbb{E}_\omega[\omega]$, respectively. Expanding their utilities around these bliss points, we can write

$$\mathbb{E}_{\omega|s}\left[U^R(y,\omega)\big|\, s\right] = -(y-\mathbb{E}_\omega[\omega|\, s])^2 - Var(\omega|\, s)$$

[14] It is easy to generalize the results to nonlinear biases; this is left to the reader.

and

$$\mathbb{E}_{\omega|s}\left[U^R(y,\omega)\middle|\, s\right] = -\left(y - 2\mathbb{E}_\omega\left[\omega|\, s\right] + \mathbb{E}_\omega\left[\omega\right]\right)^2 - 4Var\left(\omega|\, s\right),$$

where $Var(\omega|\, s)$ is the conditional variance given s. Observe that in these expressions, only the conditional expectation of ω given s interacts nontrivially with the decision y. Other details of the distribution, that is the variance, affect the level of utility but do not impact on the optimal choice in each state. This suggests a change of variables, reformulating the model in terms of communicating about what is believed to be optimal, rather than about signals themselves.

Let $z(s)$ denote the density of s and let $q(s) \equiv \mathbb{E}_\omega\left[\omega|\, s\right]$ denote the conditional expectation function, and suppose this function is strictly increasing and differentiable in s. Before the signal s is realized, the value that the function $q(s)$ takes is random; let θ denote this random variable. Since for any $\hat{s}$ and $\hat{\theta} = q(\hat{s})$, $\Pr\left[s \leq \hat{s}\right] = \Pr\left[q(s) \leq q(\hat{s})\right] = \Pr\left[s \leq q^{-1}\left(\hat{\theta}\right)\right]$, I have

$$f(\theta) = z\left(q^{-1}(\theta)\right)\frac{1}{q_s\left(q^{-1}(\theta)\right)},$$

where $q^{-1}(\cdot)$ denotes the inverse of the function q. Totally differentiating $\theta = q(s)$ and rearranging, I have $\frac{1}{q_s(s)}d\theta = ds$. Thus, letting $\underline{\theta} \equiv q(\underline{s})$ and $\overline{\theta} \equiv q(\overline{s})$, I can write

$$\mathbb{E}_{\omega,s}U^R(y,\omega) = -\int_{\underline{\theta}}^{\overline{\theta}} (y-\theta)^2 f(\theta)\, d\theta - \mathbb{E}_s Var\left(\omega|\, s\right) \tag{20}$$

and

$$\mathbb{E}_{\omega,s}U^S(y,\omega) = -\int_{\underline{\theta}}^{\overline{\theta}} \left(y - 2\theta + \mathbb{E}_\omega\left[\omega\right]\right)^2 f(\theta)\, d\theta - 4\mathbb{E}_s Var\left(\omega|\, s\right). \tag{21}$$

(20) and (21) are representations of the communication problem that are equivalent to the original one, but much more useful. These representations make clear that the equilibrium effects of changes in the quality of information depend on how the ex ante distribution of the conditional expectation, θ, depends on the quality of information. Let the distribution with density $g(\theta)$ and cdf $G(\theta)$ denote an alternative information structure. When is it true that the alternative distribution of the conditional expectation corresponds to better information? Blackwell (1951) provides an ordering requiring that the distribution of the posterior under

the more informative information structure should be more risky in the sense of second order stochastic dominance (SOSD) than under the less informative information structure. Since θ is the first moment of the posterior, we require consistently with this notion that $\int_{\underline{\theta}}^{\theta} G(\tau)\,d\tau \geq \int_{\underline{\theta}}^{\theta} F(\tau)\,d\tau$ for all θ. By the law of iterated expectations, it is always true that $\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)\,d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \theta g(\theta)\,d\theta = \mathbb{E}_{\omega}[\omega]$, implying for distributions with the same support that $\int_{\underline{\theta}}^{\overline{\theta}} G(\theta)\,d\theta = \int_{\underline{\theta}}^{\overline{\theta}} F(\theta)\,d\theta$. Obviously, this ordering provides too little structure to obtain clear comparative statics results. Szalay (2009) imposes more structure. Let $\tilde{\theta} = q(\tilde{s})$ denote the conditional expectation arising from a signal equal to its expected value, $\tilde{s}$. Szalay (2009) assumes that

$$G(\theta) \gtreqqless F(\theta) \text{ for } \theta \lesseqqgtr \tilde{\theta}. \tag{22}$$

(22) is a mean reverting First Order Stochastic Dominance condition, implying that the distribution g is more risky in the sense of a mean preserving spread than the distribution f. While stronger than the Blackwell ordering, Lemma 2 and Theorem 3 suggest that the ordering is still not strong enough for comparative statics purposes. A still stronger ordering is obtained if we require that for $\theta' < \theta''$

$$g(\theta'')\,f(\theta') \leq (\geq)\, g(\theta')\,f(\theta'') \text{ for } \tilde{\theta} \geq \theta'' \ (\tilde{\theta} \leq \theta'). \tag{23}$$

(23) is stronger than (22) in the sense that for two distributions with $F\left(\tilde{\theta}\right) = G\left(\tilde{\theta}\right)$ (which is natural for symmetric distributions), (23) implies (22). Note that (23) is precisely the mean reverting monotone likelihood ratio property introduced in Theorem 3, allowing for the case that distribution g has wider support than distribution f. Hence, the following proposition is now obvious:

Proposition 1 *Suppose that f and g are symmetric and logconcave. Moreover, suppose that the distribution with density g corresponds to better information than the one with density f in the sense of* (23)*. Then, for any finite number of induced Receiver choices, N, the unique equilibrium of the communication game is more dispersed in the sense of Theorem 3.*

The proof of this proposition follows directly from Theorem 3 and the discussion preceding the proposition.

Proposition 1 should prove useful to analyze information acquisition in a general way. As the discussion preceding the Proposition has shown, the ranking of information structures is consistent with other rankings discussed in the literature. The most general treatment of orders on the distribution of the conditional expectation is given by Ganuza and Penalva (2010). On top of the concepts already mentioned, they study distributions that are comparable in the convex order and the dispersive order. As noted above the means of θ under distribution f and g, respectively, are identical. Moroever, for distributions with the same mean, distribution g is higher in the convex order than f if and only if distribution g is a mean preserving spread of f. Therefore, (23) also implies that distribution g is higher in the convex order than distribution f.[15]

To illustrate the features of the solution, consider the following example. Suppose the distribution of s is uniform on $[0,1]$ and suppose, $\phi(\omega|\, s)$, the conditional density of ω given s takes the form $\phi(\omega|\, s) = 1 + \alpha\left(\omega - \frac{1}{2}\right)\left(s - \frac{1}{2}\right)$, where $\alpha \in (0,4)$ is a parameter that measures the informativeness of the signal s. Then, $q(s) = \frac{1}{2} + \frac{\alpha}{12}\left(s - \frac{1}{2}\right)$ and $f(\theta) = \frac{12}{\alpha}$. Take now $\alpha \in \{\underline{\alpha}, \overline{\alpha}\}$ where $\underline{\alpha} < \overline{\alpha}$. For any α, the distribution of θ is uniform. For the more informative experiment where $\alpha = \overline{\alpha}$, the distribution of θ has a wider support. Let $a_1^3(\alpha)$ and $a_1^3(\alpha)$ denote the interior thresholds of the unique three partition equilibrium for given α. It is easy to verify that these values are $a_1^3(\alpha) = \frac{1}{2} - \frac{1}{168}\alpha$ and $a_2^3(\alpha) = \frac{1}{2} + \frac{1}{168}\alpha$, respectively, so the solution is indeed more spread around the mean $\frac{1}{2}$ the higher is α. While this is a nicely tractable example, it should be stressed that nothing depends on the moving support feature of this example. It is easy to construct examples with a non-moving support.[16]

7 Conclusion

This paper studies the role of the distribution of types in the strategic information transmission game. I show that logconcavity of the density combined with a restriction on the

[15] There is no connection between the conditions studied here and the dispersive order.

[16] The interested reader is referred to Szalay (2009). The example given there needs to be amended in an obvious way so as to make it work for the support $[0,1]$ of ω assumed here. This is left to the reader.

Receiver's utility, whereby this restriction is independent of the distributional assumption, implies uniqueness of equilibria inducing a given number of Receiver choices. Moreover, I provide comparative statics for distributions that can be ordered by their likelihood ratios, showing in particular that equilibria in a communication game where the Sender has better information are more spread out than when the Sender's information is worse. I sketch a general model to study improvements in the quality of information, which hopefully proves useful in future applications.

While this work focusses entirely on the positive effects of varying the quality of information, in companion work (Esö and Szalay (2012)) we explore the normative aspects of varying the quality of information. Among other things, we study incentives for information acquisition as a function of the informativeness of equilibrium communication.

8 Appendix

Proof of Lemma 2. Suppose that $\frac{g}{f}$ is strictly increasing in ω for all $\omega \in [0,1]$. The first-order condition for $y(\underline{x}, \overline{x}; f)$ is

$$\int_{\underline{x}}^{\overline{x}} U_1^R\left(y\left(\underline{x}, \overline{x}; f\right), \omega\right) f(\omega)\, d\omega = 0$$

To prove that $y(\underline{x}, \overline{x}; g) > y(\underline{x}, \overline{x}; f)$, it suffices to show that

$$\int_{\underline{x}}^{\overline{x}} U_1^R\left(y\left(\underline{x}, \overline{x}; f\right), \omega\right) g(\omega)\, d\omega > 0.$$

Indeed, multiplying and dividing by $\frac{f(\omega)}{F(\overline{x})-F(\underline{x})}$, I have

$$\begin{aligned}
&\left(F(\overline{x}) - F(\underline{x})\right) \int_{\underline{x}}^{\overline{x}} U_1^R\left(y\left(\underline{x}, \overline{x}; f\right), \omega\right) \frac{g(\omega)}{f(\omega)} \frac{f(\omega)}{F(\overline{x}) - F(\underline{x})} d\omega \\
= \;& \left(F(\overline{x}) - F(\underline{x})\right) Cov\left(U_1^R\left(y\left(\underline{x}, \overline{x}; f\right), \omega\right), \left.\frac{g(\omega)}{f(\omega)}\right| \omega \in [\underline{x}, \overline{x}]\right),
\end{aligned}$$

where the equality uses the fact that $\int_{\underline{x}}^{\overline{x}} U_1^R\left(y\left(\underline{x},\overline{x};f\right),\omega\right) f\left(\omega\right) d\omega = 0$. Now, since $U_{12}^R\left(y,\omega\right) > 0$ and $\frac{g(\omega)}{f(\omega)}$ is increasing in ω, I have indeed

$$Cov\left(U_1^R\left(y\left(\underline{x},\overline{x};f\right),\omega\right), \left.\frac{g\left(\omega\right)}{f\left(\omega\right)}\right| \omega\in\left[\underline{x},\overline{x}\right]\right) > 0.$$

By concavity of the function $U^R\left(y,\omega\right)$ in y, y needs to be increased when the density is changed from f to g. So, we have shown that $\frac{g}{f}$ being strictly increasing in ω for all $\omega\in[0,1]$ implies that

$$y\left(\underline{x},\overline{x};g\right) > y\left(\underline{x},\overline{x};f\right).$$

To see the converse is also true, suppose that there is an interval $[\underline{x},\overline{x}]$ such that $\frac{g(\omega)}{f(\omega)}$ is non-increasing over that interval. This implies that

$$Cov\left(U_1^R\left(y\left(\underline{x},\overline{x};f\right),\omega\right), \left.\frac{g\left(\omega\right)}{f\left(\omega\right)}\right| \omega\in\left[\underline{x},\overline{x}\right]\right) \leq 0$$

and thus that $y\left(\underline{x},\overline{x};g\right) \leq y\left(\underline{x},\overline{x};f\right)$. ■

Proof of Lemma 3. Consider two intervals of the same length, $[\underline{x},\overline{x}]$ and $[\underline{x}+\Delta,\overline{x}+\Delta]$. Let $y^* = y\left(\underline{x},\overline{x}\right)$ solve

$$\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*,\omega\right) f\left(\omega\right) d\omega = 0$$

and let $y^*\left(\Delta\right) = y\left(\underline{x}+\Delta,\overline{x}+\Delta\right)$ solve

$$\int_{\underline{x}+\Delta}^{\overline{x}+\Delta} U_1^R\left(y^*\left(\Delta\right),\omega\right) f\left(\omega\right) d\omega = 0.$$

We wish to show that $y^*\left(\Delta\right) \leq y^* + \Delta$. By $U_{11}^R\left(y,\omega\right) < 0$, this is equivalent to

$$\int_{\underline{x}+\Delta}^{\overline{x}+\Delta} U_1^R\left(y^*+\Delta,\omega\right) \frac{f\left(\omega\right)}{F\left(\overline{x}+\Delta\right) - F\left(\underline{x}+\Delta\right)} d\omega \leq 0.$$

Notice that

$$\int_{\underline{x}+\Delta}^{\overline{x}+\Delta} U_1^R\left(y^*+\Delta,\omega\right)\frac{f\left(\omega\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}d\omega=\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\frac{f\left(\omega+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}d\omega.$$

Moreover,

$$\begin{aligned}
&\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\frac{f\left(\omega+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}d\omega\\
=\;&\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\left(\frac{f\left(\omega+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}-\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}\right)d\omega\\
&+\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}d\omega
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}d\omega\\
=\;&\int_{\underline{x}}^{\overline{x}}\left[U_1^R\left(y^*,\omega\right)+\int_0^{\Delta}\frac{\partial}{\partial z}U_1^R\left(y^*+z,\omega+z\right)dz\right]\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}d\omega\\
=\;&\int_{\underline{x}}^{\overline{x}}\left[\int_0^{\Delta}\frac{\partial}{\partial z}U_1^R\left(y^*+z,\omega+z\right)dz\right]\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}d\omega,
\end{aligned}$$

where the second line follows from the first-order condition defining y^*. By assumption $\frac{\partial}{\partial z}U_1^R\left(y^*+z,\omega+z\right)\leq 0$; thus the expression is maximized for a utility function that satisfies $\frac{\partial}{\partial z}U_1^R\left(y^*+z,\omega+z\right)=0$, in which case it takes value zero. Hence, the overall expression is nonpositive.

Consider now the expression

$$\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\left(\frac{f\left(\omega+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}-\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}\right)d\omega.$$

After an integration by parts, we have

$$\int_{\underline{x}}^{\overline{x}} U_1^R\left(y^*+\Delta,\omega+\Delta\right)\left(\frac{f\left(\omega+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}-\frac{f\left(\omega\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}\right)d\omega$$

$$= \quad -\int_{\underline{x}}^{\overline{x}} U_{12}^R\left(y^*+\Delta,\omega+\Delta\right)\left(\frac{F\left(\omega+\Delta\right)-F\left(\underline{x}+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)}-\frac{F\left(\omega\right)-F\left(\underline{x}\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}\right)d\omega.$$

By $U_{12}^R\left(y,\omega\right) \geq 0$, $U_1^R\left(y^*+\Delta,\omega+\Delta\right)$ is an increasing function of ω. Thus, the effect is nonpositive if

$$\frac{F\left(\omega+\Delta\right)-F\left(\underline{x}+\Delta\right)}{F\left(\overline{x}+\Delta\right)-F\left(\underline{x}+\Delta\right)} \geq \frac{F\left(\omega\right)-F\left(\underline{x}\right)}{F\left(\overline{x}\right)-F\left(\underline{x}\right)}, \tag{24}$$

that is if the truncated distribution of $\omega \in [\underline{x},\overline{x}]$ dominates the truncated distribution of $\omega \in [\underline{x}+\Delta,\overline{x}+\Delta]$ in the sense of First Order Stochastic Dominance.

As argued in the text, a fully supported density $f\left(\omega\right)$ is logconcave if and only if for $\Delta > 0$, $\frac{f(\omega)}{f(\omega+\Delta)}$ is nondecreasing in ω. Hence, for a logconcave density

$$\frac{\partial}{\partial\omega}\frac{\frac{f(\omega)}{F(\overline{x})-F(\underline{x})}}{\frac{f(\omega+\Delta)}{F(\overline{x}+\Delta)-F(\underline{x}+\Delta)}} \geq 0. \tag{25}$$

By Milgrom (1981), the monotone likelihood ratio property, (25) , implies First Order Stochastic Dominance, (24) . This establishes sufficiency of the conditions.

To see the converse statement, suppose there is an interval $[\underline{x},\overline{x}]$ over which $\frac{f(\omega)}{f(\omega+\Delta)}$ is decreasing in ω. Suppose we choose in addition a utility function such that $\frac{\partial}{\partial z}U_1^R\left(y+z,\omega+z\right) = 0$. Then, both inequalities in conditions (25) and (24) are reversed, and in fact hold as strict inequalities. Thus, for this construction we would have $y^*\left(\Delta\right) > y^*+\Delta$. ■

Proof of Theorem 1. Recall that, for all Receiver utility functions such that $\frac{dy^R(\omega)}{d\omega} \leq 1$, logconcavity of $f\left(\omega\right)$ is equivalent to

$$\frac{\partial}{\partial\underline{x}}y_i\left(\underline{x},\overline{x}\right)+\frac{\partial}{\partial\overline{x}}y_i\left(\underline{x},\overline{x}\right) \leq 1 \tag{26}$$

for for all $\underline{x},\overline{x}$ such that $0 \leq \underline{x} < \overline{x} \leq 1$.

Suppose there is an equilibrium inducing N distinct Receiver actions, i.e., the system (2) has a solution with initial condition a_0^N and given end point $a_N^N = 1$. It is useful to split (2) into two sets. For arbitrary initial condition $x \in [0, 1)$ split the conditions into

$$U^S\left(y_1\left(a_0^N, x\right), x\right) = U^S\left(y_2\left(x, a_2^N(x)\right), x\right) \tag{27}$$

on the one hand and

$$U^S\left(y_2\left(x, a_2^N(x)\right), a_2^N(x)\right) = U^S\left(y_3\left(a_2^N(x), a_3^N(x)\right), a_2^N(x)\right) \tag{28}$$

$$U^S\left(y_i\left(a_{i-1}^N(x), a_i^N(x)\right), a_i^N(x)\right) = U^S\left(y_{i+1}\left(a_i^N(x), a_{i+1}^N(x)\right), a_i^N(x)\right) \tag{29}$$

for $i = 3, \ldots, N-2$, and

$$U^S\left(y_{N-1}\left(a_{N-2}^N(x), a_{N-1}^N(x)\right), a_{N-1}^N(x)\right) = U^S\left(y_N\left(a_{N-1}^N(x), a_N^N\right), a_{N-1}^N(x)\right) \tag{30}$$

on the other hand.

x can be viewed as an initial condition to the system given by (28), (29), and (30). When we require x to satisfy (27) as well, then x is forced to take its equilibrium value. To prove the result, we need to show that (27) has exactly one solution in x. Differentiating (27), we find that the effect of a marginal increase in x on the left-hand side is

$$U_1^S\left(y_1\left(a_0^N, x\right), x\right) \frac{\partial y_1\left(a_0^N, x\right)}{\partial x} + U_2^S\left(y_1\left(a_0^N, x\right), x\right)$$

while the effect of a marginal increase in x on the right-hand side is

$$U_1^S\left(y_2\left(x, a_2^N(x)\right), x\right)\left(\frac{\partial y_2\left(x, a_2^N(x)\right)}{\partial x} + \frac{\partial y_2\left(x, a_2^N(x)\right)}{\partial a_2^N(x)} \frac{da_2^N(x)}{dx}\right) + U_2^S\left(y_2\left(x, a_2^N(x)\right), x\right)$$

I now show that the latter effect always dominates the former, using the implications of logconcavity from the Lemma and showing that $\frac{da_2^N(x)}{dx} \in (0, 1)$. In turn, to show that $\frac{da_2^N(x)}{dx} \in (0, 1)$, I need to show how the entire sequence $\left\{a_2^N(x), \ldots, a_{N-1}^N(x)\right\}$ changes with the initial condition x.

In what follows I suppress the dependence of y_i on its arguments where this is needed for reasons of space.

By smoothness of the Receiver's optimal choice as a function of the threshold values, the system's solution varies smoothly in the initial condition x. Differentiating (30) totally with respect to a_{N-2}^N and a_{N-1}^N we obtain

$$\frac{da_{N-1}^N}{da_{N-2}^N} = \frac{U_1^S\left(y_{N-1}, a_{N-1}^N\right)\frac{\partial y_{N-1}}{\partial a_{N-2}^N}}{U_1^S\left(y_N, a_{N-1}^N\right)\frac{\partial y_N}{\partial a_{N-1}^N} + U_2^S\left(y_N, a_{N-1}^N\right) - U_1^S\left(y_{N-1}, a_{N-1}^N\right)\frac{\partial y_{N-1}}{\partial a_{N-1}^N} - U_2^S\left(y_{N-1}, a_{N-1}^N\right)}. \tag{31}$$

The Sender's indifference condition, $U^S\left(y_{N-1}, a_{N-1}^N\right) = U^S\left(y_N, a_{N-1}^N\right)$, implies that $y_{N-1}\left(a_{N-2}^N, a_{N-1}^N\right) < y^S\left(a_{N-1}^N\right) < y_N\left(a_{N-1}^N, a_N^N\right)$ and thus that $U_1^S\left(y_N, a_{N-1}^N\right) < 0 < U_1^S\left(y_{N-1}, a_{N-1}^N\right)$. Therefore, the numerator is positive. The denominator is positive iff

$$U_1^S\left(y_N, a_{N-1}^N\right)\frac{\partial y_N}{\partial a_{N-1}^N} + U_2^S\left(y_N, a_{N-1}^N\right) > U_1^S\left(y_{N-1}, a_{N-1}^N\right)\frac{\partial y_{N-1}}{\partial a_{N-1}^N} + U_2^S\left(y_{N-1}, a_{N-1}^N\right).$$

By the now familiar arguments, the left-hand side is bounded below by and strictly larger than $U_1^S\left(y_N, a_{N-1}^N\right) + U_2^S\left(y_N, a_{N-1}^N\right)$, while the right-hand side is bounded above by and strictly smaller than $U_1^S\left(y_{N-1}, a_{N-1}^N\right) + U_2^S\left(y_{N-1}, a_{N-1}^N\right)$. So, for a function such that $U_1^S(y,a) + U_2^S(y,a)$ is nondecreasing in y, the denominator is positive, so $\frac{da_{N-1}^N}{da_{N-2}^N} > 0$. Using that the denominator is positive, I have $\frac{da_{N-1}^N}{da_{N-2}^N} < 1$ iff

$$U_1^S\left(y_{N-1}, a_{N-1}^N\right)\left(\frac{\partial y_{N-1}}{\partial a_{N-2}^N} + \frac{\partial y_{N-1}}{\partial a_{N-1}^N}\right) + U_2^S\left(y_{N-1}, a_{N-1}^N\right) < U_1^S\left(y_N, a_{N-1}^N\right)\frac{\partial y_N}{\partial a_{N-1}^N} + U_2^S\left(y_N, a_{N-1}^N\right).$$

The left-hand side is weakly smaller than $U_1^S\left(y_{N-1}, a_{N-1}^N\right) + U_2^S\left(y_{N-1}, a_{N-1}^N\right)$ while the right-hand side is strictly larger than $U_1^S\left(y_N, a_{N-1}^N\right) + U_2^S\left(y_N, a_{N-1}^N\right)$. So, for a function such that $U_1^S(y,a) + U_2^S(y,a)$ is nondecreasing in y, I have $\frac{da_{N-1}^N}{da_{N-2}^N} < 1$.

Differentiating (29) totally, and substituting for

$$\frac{da_{i-1}^N}{da_{i+1}^N} = \frac{da_{i-1}^N}{da_i^N}\frac{da_i^N}{da_{i+1}^N},$$

I have

$$\frac{da_i^N}{da_{i-1}^N} = \frac{U_1^S\left(y_i, a_i^N\right)\frac{\partial y_i}{\partial a_{i-1}^N}}{U_1^S\left(y_{i+1}, a_i^N\right)\left(\frac{\partial y_{i+1}}{\partial a_i^N} + \frac{\partial y_{i+1}}{\partial a_{i+1}^N}\frac{da_{i+1}^N}{da_i^N}\right) + U_2^S\left(y_{i+1}, a_i^N\right) - U_1^S\left(y_i, a_i^N\right)\frac{\partial y_i}{\partial a_i^N} - U_2^S\left(y_i, a_i^N\right)}. \tag{32}$$

I can now show by induction that $\frac{da_{i+1}^N}{da_i^N} \in (0,1)$ implies that $\frac{da_i^N}{da_{i-1}^N} \in (0,1)$.

By the Sender's indifference condition, I have $y_i\left(a_{i-1}^N, a_i^N\right) < y^S\left(a_i^N\right) < y_{i+1}\left(a_i^N, a_{i+1}^N\right)$, and thus $U_1^S\left(y_i\left(a_{i-1}^N, a_i^N\right), a_i^N\right) > 0 > U_1^S\left(y_{i+1}\left(a_i^N, a_{i+1}^N\right), a_i^N\right)$. Hence, the numerator on the right-hand side of (32) is positive. The denominator on the right hand side of (32) is positive iff

$$U_1^S\left(y_{i+1}, a_i^N\right)\left(\frac{\partial y_{i+1}}{\partial a_i^N} + \frac{\partial y_{i+1}}{\partial a_{i+1}^N}\frac{da_{i+1}^N}{da_i^N}\right) + U_2^S\left(y_{i+1}, a_i^N\right) > U_1^S\left(y_i, a_i^N\right)\frac{\partial y_i}{\partial a_i^N} + U_2^S\left(y_i, a_i^N\right)$$

Suppose that $\frac{da_{i+1}^N}{da_i^N} \in (0,1)$. Then, I have

$$\frac{\partial y_{i+1}\left(a_i^N, a_{i+1}^N\right)}{\partial a_i^N} + \frac{\partial y_{i+1}\left(a_i^N, a_{i+1}^N\right)}{\partial a_{i+1}^N}\frac{da_{i+1}^N}{da_i^N} < 1$$

and thus that the left-hand side is strictly larger than $U_1^S\left(y_{i+1}, a_i^N\right) + U_2^S\left(y_{i+1}, a_i^N\right)$. The right-hand side is strictly smaller than $U_1^S\left(y_i, a_i^N\right) + U_2^S\left(y_i, a_i^N\right)$. Hence, for a function such that $U_1^S(y,a) + U_2^S(y,a)$ is nondecreasing in y, the denominator is positive. Hence, $\frac{da_i^N}{da_{i-1}^N} < 1$ iff

$$U_1^S\left(y_i, a_i^N\right)\left(\frac{\partial y_i}{\partial a_{i-1}^N} + \frac{\partial y_i}{\partial a_i^N}\right) + U_2^S\left(y_i, a_i^N\right) < U_1^S\left(y_{i+1}, a_i^N\right)\left(\frac{\partial y_{i+1}}{\partial a_i^N} + \frac{\partial y_{i+1}}{\partial a_{i+1}^N}\frac{da_{i+1}^N}{da_i^N}\right) + U_2^S\left(y_{i+1}, a_i^N\right). \tag{33}$$

By the now familiar argument, this holds true for a function such that $U_1^S(y,a) + U_2^S(y,a)$ is nondecreasing in y. Therefore, the inductive hypotheses, $\frac{da_{i+1}^N}{da_i^N} \in (0,1)$, implies that $\frac{da_i^N}{da_{i-1}^N} \in (0,1)$. This argument shows that $\frac{da_i^N}{da_{i-1}^N} \in (0,1)$ for $i = 3, \ldots, N-2$. Note in particular, that the argument implies that $\frac{da_3^N}{da_2^N} \in (0,1)$.

Finally, consider (28). Totally differentiating the condition and substituting for

$$\frac{da_3^N}{dx} = \frac{da_3^N}{da_2^N}\frac{da_2^N}{dx}$$

I obtain I have

$$\frac{da_2^N}{dx} = \frac{U_1^S\left(y_2, a_2^N\right)\frac{\partial y_2}{\partial x}}{U_1^S\left(y_3, a_2^N\right)\frac{\partial y_3}{\partial a_2^N} + U_1^S\left(y_3, a_2^N\right)\frac{\partial y_3}{\partial a_3^N}\frac{da_3^N}{da_2^N} + U_2^S\left(y_3, a_2^N\right) - U_1^S\left(y_2, a_2^N\right)\frac{\partial y_2}{\partial a_2^N} - U_2^S\left(y_2, a_2^N\right)}. \tag{34}$$

By the now familiar arguments $y_2\left(x,a_2^N\right)<y^S\left(a_2^N\right)<y_3\left(a_2^N,a_3^N\right)$ and hence $U_1^S\left(y_2,a_2^N\right)>0>U_1^S\left(y_3,a_2^N\right)$. Hence, (34) has got the exact same structure as (32) has, and hence I have $\frac{da_2^N}{dx}\in(0,1)$.

Hence we have shown that there is a unique x^*, such that

$$U^S\left(y_1\left(a_0^N,x\right),x\right)=U^S\left(y_2\left(x,a_2^N(x)\right),x\right) \tag{35}$$

for $x=x^*$, where $a_2^N(x)$ is determined by (28), (29), and (30).

Since N is arbitrary in this argument, we have shown that for any N, there is at most one solution to (2). ■

Proof of Theorem 2. The equilibrium of the game is the initial condition $a_0^N(h)=0$, the final condition $a_N^N(h)=1$, together with the system of equations

$$U^S\left(y_i^N(h),a_i^N(h)\right)=U^S\left(y_{i+1}^N(h),a_i^N(h)\right) \text{ for } i=1,\ldots,N-1. \tag{36}$$

for $i=1,\ldots,N-1$.

Totally differentiating (19) with respect to a_i^N and a_{i-1}^N -suppressing the dependence of these values on h for brevity - I obtain

$$\frac{da_i^N}{da_{i-1}^N}=\frac{U_1^S\left(y_i^N,a_i^N\right)\frac{\partial y_i^N}{\partial a_{i-1}^N}}{U_1^S\left(y_{i+1}^N,a_i^N\right)\frac{\partial y_{i+1}^N}{\partial a_i^N}+U_2^S\left(y_{i+1}^N,a_i^N\right)-U_1^S\left(y_i^N,a_i^N\right)\frac{\partial y_i^N}{\partial a_i^N}-U_2^S\left(y_i^N,a_i^N\right)}.$$

By the fact that $y_i^N<y^S\left(a_i^N\right)<y_{i+1}^N$, I have $U_1^S\left(y_i^N,a_i^N\right)>0>U_1^S\left(y_{i+1}^N,a_i^N\right)$. Hence, the numerator is positive. Moreover, since $\frac{\partial y_{i+1}^N}{\partial a_i^N},\frac{\partial y_i^N}{\partial a_i^N}\in(0,1)$, the denominator is strictly larger than

$$U_1^S\left(y_{i+1}^N,a_i^N\right)+U_2^S\left(y_{i+1}^N,a_i^N\right)-U_1^S\left(y_i^N,a_i^N\right)-U_2^S\left(y_i^N,a_i^N\right)>0,$$

where the conclusion follows from the fact that $U_1^S(y,a)+U_2^S(y,a)$ is nondecreasing in y. Hence, $\frac{da_i^N}{da_{i-1}^N}>0$. An identical argument can be given to show that $\frac{da_i^N}{da_{i+1}^N}>0$. So, a_i^N is increasing in a_{i-1}^N and a_{i+1}^N.

Suppose that, for any given sequence of thresholds, $y_i^N(g)>y_i^N(f)$ and $y_{i+1}^N(g)>y_{i+1}^N(f)$ for $i=1,\ldots,N-1$. The equilibrium condition for the thresholds under distribution f is

$$U^S\left(y_i^N(f),a_i^N(f)\right)=U^S\left(y_{i+1}^N(f),a_i^N(f)\right).$$

Keep now the sequence unchanged, but change the distribution from f to g and adjust the Receiver's choices accordingly to

$$y_i^N(f,g) \equiv y_i^N\left(a_{i-1}^N(f), a_i^N(f); g\right)$$

and

$$y_{i+1}^N(f,g) = y_i^N\left(a_i^N(f), a_{i+1}^N(f); g\right)$$

To prove that $a_i^N(g) > a_i^N(f)$, we need to show that

$$U^S\left(y_i^N(f,g), a_i^N(f)\right) - U^S\left(y_{i+1}^N(f,g), a_i^N(f)\right) > 0.$$

To show this, we first establish that $y_i^N(f,g) < y^S\left(a_i^N(f)\right) < y_{i+1}^N(f,g)$. Since $y_{i+1}^N(f,g) > y_{i+1}^N(f)$ and $y_{i+1}^N(f) > y^S\left(a_i^N(f)\right)$, the second inequality is trivially satisfied. So, it suffices to show that $y_i^N(f,g) < y^S\left(a_i^N(f)\right)$. For $b(\omega) > 0$, we have $y^R\left(a_i^N(f)\right) < y^S\left(a_i^N(f)\right)$. For any $a_{i-1}^N < a_i^N$, we have $y_i^N\left(a_{i-1}^N, a_i^N\right) < y^R\left(a_i^N\right)$. Therefore, it follows that $y_i^N(f,g) < y^S\left(a_i^N(f)\right)$. Thus, we have shown that $U_1^S\left(y, a_i^N(f)\right) > 0$ for all $y \in \left[y_i^N(f), y_i^N(f,g)\right]$ and $U_1^S\left(y, a_i^N(f)\right) < 0$ for all $y \in \left[y_{i+1}^N(f), y_{i+1}^N(f,g)\right]$. Notice that

$$\begin{aligned}
& U^S\left(y_i^N(f,g), a_i^N(f)\right) - U^S\left(y_{i+1}^N(f,g), a_i^N(f)\right) \\
= \; & U^S\left(y_i^N(f,g), a_i^N(f)\right) - U^S\left(y_{i+1}^N(f,g), a_i^N(f)\right) - \left(U^S\left(y_i^N(f), a_i^N(f)\right) - U^S\left(y_{i+1}^N(f), a_i^N(f)\right)\right) \\
= \; & \int_{y_i^N(f)}^{y_i^N(f,g)} U_1^S\left(y, a_i^N(f)\right) dy - \int_{y_{i+1}^N(f)}^{y_{i+1}^N(f,g)} U_1^S\left(y, a_i^N(f)\right) dy.
\end{aligned}$$

The first equality follows from the fact that $U^S\left(y_i^N(f), a_i^N(f)\right) - U^S\left(y_{i+1}^N(f), a_i^N(f)\right) = 0$, the second from the fundamental theorem of differentiation. By the arguments just made, we have

$$\int_{y_i^N(f)}^{y_i^N(f,g)} U_1^S\left(y, a_i^N(f)\right) dy - \int_{y_{i+1}^N(f)}^{y_{i+1}^N(f,g)} U_1^S\left(y, a_i^N(f)\right) dy > 0.$$

The next step is to show that $U^S\left(y_i^N(f,g), a_i^N(f)\right) - U^S\left(y_{i+1}^N(f,g), a_i^N(f)\right)$ is decreasing in $a_i^N(f)$. Differentiating the difference, we have

$$
\begin{aligned}
& \frac{\partial}{\partial a_i^N(f)}\left[U^S\left(y_i^N(f,g),a_i^N(f)\right)-U^S\left(y_{i+1}^N(f,g),a_i^N(f)\right)\right] \\
= \; & U_1^S\left(y_i^N(f,g),a_i^N(f)\right)\frac{\partial y_i^N(f,g)}{\partial a_i^N(f)}+U_2^S\left(y_i^N(f,g),a_i^N(f)\right) \\
& -U_1^S\left(y_{i+1}^N(f,g),a_i^N(f)\right)\frac{\partial y_{i+1}^N(f,g)}{\partial a_i^N(f)}-U_2^S\left(y_{i+1}^N(f,g),a_i^N(f)\right)
\end{aligned}
$$

By the now familiar arguments, we have

$$
\begin{aligned}
& \frac{\partial}{\partial a_i^N(f)}\left[U^S\left(y_i^N(f,g),a_i^N(f)\right)-U^S\left(y_{i+1}^N(f,g),a_i^N(f)\right)\right] \\
< \; & U_1^S\left(y_i^N(f,g),a_i^N(f)\right)+U_2^S\left(y_i^N(f,g),a_i^N(f)\right)-U_1^S\left(y_{i+1}^N(f,g),a_i^N(f)\right)-U_2^S\left(y_{i+1}^N(f,g),a_i^N(f)\right) \\
\leq \; & 0,
\end{aligned}
$$

where the last inequality uses that $U_1^S(y,a)+U_2^S(y,a)$ is nondecresing in y. So, $U^S\left(y_i^N(f,g),a_i^N(f)\right)-U^S\left(y_{i+1}^N(f,g),a_i^N(f)\right)$ is indeed decreasing in $a_i^N(f)$. Hence, to reset the difference equal to zero, we need to increase $a_i^N(f)$. Hence, $a_i^N(g)>a_i^N(f)$. ■

Proof of Theorem 3. To prove the theorem, I construct symmetric equilibria and show that the equilibrium thresholds in such symmetric equilibria are more spread out for distribution g.

Part i: symmetric equilibria.

There are two cases to distinguish: a) N is even and b) N is odd. I provide a detailed proof of case a and sketch the argument for case b.

a) Suppose N is even. In a symmetric equilibrium, $a_{\frac{N}{2}}^N(h)=.5$ for $h=f,g$. Consider now the half-support for $\omega\leq .5$ and ignore the thresholds $a_i^N(h)$ for $i>\frac{N}{2}$ for the time being. In what follows I suppress the dependence of the thresholds on the distribution where this can be done without creating confusion. Define $\omega_i^N=\mathbb{E}\left[\omega|\omega\in\left[a_{i-1}^N,a_i^N\right]\right]$ and $\omega_{i+1}^N=\mathbb{E}\left[\omega|\omega\in\left[a_i^N,a_{i+1}^N\right]\right]$. Suppose the thresholds a_i^N for $i\leq\frac{N}{2}$ are a solution to (2) with initial condition $a_0^N=0$ and final condition $a_{\frac{N}{2}}^N(h)=.5$, thus - using the fact that utilities are quadratic -

$$\left(a_i^N+b\left(a_i^N\right)\right)-\omega_i^N=\omega_{i+1}^N-\left(a_i^N+b\left(a_i^N\right)\right)$$

for $i = 1, \ldots, \frac{N}{2} - 1$. Letting $\delta_i \equiv .5 - a_i^N$, I can write

$$(.5 - \delta_i + b\,(.5 - \delta_i)) - \omega_i^N = \omega_{i+1}^N - (.5 - \delta_i + b\,(.5 - \delta_i))$$

for $i = 1, \ldots, \frac{N}{2} - 1$. Rearranging, I have

$$(-\delta_i + b\,(.5 - \delta_i)) + .5 - \omega_i^N + .5 - \omega_{i+1}^N = -\,(-\delta_i + b\,(.5 - \delta_i)) \tag{37}$$

Let now $j\,(i) = \frac{N}{2} + \left(\frac{N}{2} - i\right) = N - i$ and $\delta_j \equiv a_j^N - .5$. We wish to show that (37) implies that

$$\left(a_{j(i)}^N + b\left(a_{j(i)}^N\right)\right) - \omega_{j(i)}^N = \omega_{j(i)+1}^N - \left(a_{j(i)}^N + b\left(a_{j(i)}^N\right)\right)$$

for $i = 1, \ldots, \frac{N}{2} - 1$ describes an equilibrium sequence of thresholds on the upper half-support. Substituting for δ_j

$$(\delta_j + .5 + b\,(\delta_j + .5)) - \omega_j^N = \omega_{j+1}^N - (\delta_j + .5 + b\,(\delta_j + .5))$$

and rearranging

$$(\delta_j + b\,(\delta_j + .5)) = \omega_j^N - .5 + \omega_{j+1}^N - .5 - (\delta_j + b\,(\delta_j + .5))\,. \tag{38}$$

Clearly, for $\delta_i = \delta_j$, I have $.5 - \omega_i^N = \omega_j^N - .5$ and $.5 - \omega_{i+1}^N = \omega_{j+1}^N - .5$ by symmetry of the distribution. Moreover, (38) characterizes an equilibrium sequence of thresholds for the upper half support iff

$$2\delta_i - 2b\,(.5 - \delta_i) = 2\delta_j + 2b\,(\delta_j + .5)\,,$$

which is satisfied since $b\,(.5 - \delta) = b\,(\delta + .5)$.

b) N is odd. In this case I take $a_{\frac{N+1}{2}-1}^N$ as a given final condition on the lower half of the support and $a_{\frac{N+1}{2}}^N$ as a given initial condition on the upper half of the support. The equilibrium construction works in two steps. In the first step, $a_{\frac{N+1}{2}-1}^N$ and $a_{\frac{N+1}{2}}^N$ are arbitrary but for the requirement that $a_{\frac{N+1}{2}}^N - .5 = .5 - a_{\frac{N+1}{2}-1}^N$. Given this restriction, I can apply the argument of part a) to the supports $\left[0, a_{\frac{N+1}{2}-1}^N\right]$ and $\left[a_{\frac{N+1}{2}}^N, 1\right]$. In the second step, the distance $a_{\frac{N+1}{2}}^N - .5$ is adjusted to make the entire construction an equilibrium. This establishes that, given uniqueness, the equilibrium must be symmetric.

Part ii: equilibria are more spread out under distribution g.

Consider first the case where N is even. I can apply Theorem 2 to the half-supports $\left[0, \frac{1}{2}\right]$, since for any symmetric equilibrium with N even I have $a^N_{\frac{N}{2}}(h) = .5$. It follows that $a^N_i(g) < a^N_i(f)$ for $i < \frac{N}{2}$. The properties on the upper half-support follow from the fact that the distribution and bias are symmetric around the mean, thus $a^N_i(g) > a^N_i(f)$ for $i > \frac{N}{2}$.

For the case where N is odd, consider first the supports $\left[0, a^N_{\frac{N+1}{2}-1}(f)\right]$ and $\left[a^N_{\frac{N+1}{2}}(f), 1\right]$ where the bounds $a^N_{\frac{N+1}{2}-1}(f)$ and $a^N_{\frac{N+1}{2}}(f)$ are the equilibrium thresholds for the distribution f. Note that theses thresholds satisfy t $a^N_{\frac{N+1}{2}}(f) - .5 = .5 - a^N_{\frac{N+1}{2}-1}(f)$. Applying Theorem 2 to these supports, I find that for given "final condition" $a^N_{\frac{N+1}{2}-1}(f)$, changing the distribution from f to g, shifts all the conditional means $i = 1, \ldots, \frac{N+1}{2} - 1$ and all the threshold types $i = 1, \ldots, \frac{N+1}{2} - 2$ downwards. By symmetry, all conditional means and thresholds types $i = \frac{N+1}{2} + 1, \ldots, N-1$ are shifted upwards for given $a^N_{\frac{N+1}{2}}(f)$. Note that the truncated mean over the interval $\left[a^N_{\frac{N+1}{2}-1}(f), a^N_{\frac{N+1}{2}}(f)\right]$ is by symmetry equal to .5.

By the fact that the initial construction is an equilibrium under distribution f, I have

$$\left(a^N_{\frac{N+1}{2}-1}(f) + b\left(a^N_{\frac{N+1}{2}-1}(f)\right)\right) - \omega^N_{\frac{N+1}{2}-1}(f) = .5 - \left(a^N_{\frac{N+1}{2}-1}(f) + b\left(a^N_{\frac{N+1}{2}-1}(f)\right)\right),$$

where I have substituted $\omega^N_{\frac{N+1}{2}} = .5$. Changing the distribution to g, I have

$$\left(a^N_{\frac{N+1}{2}-1}(f) + b\left(a^N_{\frac{N+1}{2}-1}(f)\right)\right) - \omega^N_{\frac{N+1}{2}-1}(g) > .5 - \left(a^N_{\frac{N+1}{2}-1}(f) + b\left(a^N_{\frac{N+1}{2}-1}(f)\right)\right),$$

since $\omega^N_{\frac{N+1}{2}-1}(g) < \omega^N_{\frac{N+1}{2}-1}(f)$ by the argument made above. By logconcavity, the left-hand side is non-decreasing in $a^N_{\frac{N+1}{2}-1}$; the right-hand side is decreasing in $a^N_{\frac{N+1}{2}-1}$. Hence, decreasing $a^N_{\frac{N+1}{2}-1}$ decreases the left-hand side and increases the right-hand side, so $a^N_{\frac{N+1}{2}-1}(g) < a^N_{\frac{N+1}{2}-1}(f)$. ■

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